CS 240 – Data Structures and Data Management

Module 8: Range-Searching in Dictionaries for Points

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Based on lecture notes by many previous cs240 instructors

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Outline

8 Range-Searching in Dictionaries for Points
- Range Searches
- Multi-Dimensional Data
- Quadtrees
- kd-Trees
- Range Trees
- Conclusion

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8. Range-Searching in Dictionaries for Points
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   - Multi-Dimensional Data
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   - Range Trees
   - Conclusion

Range searches

- So far: *search*(k) looks for *one* specific item.
- New operation **range-search**: look for *all* items that fall within a given range.
  - Input: A **range**, i.e., an interval $Q = (x, x')$
    It may be open or closed at the ends.
  - Want: Report all KVPs in the dictionary whose key k satisfies $k \in Q$

Example:

5	10	11	17	19	33	45	51	55	59

range-search((18,45]) should return $\{19, 33, 45\}$

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- As usual n denotes the number of input-items.
- Let s be the **output-size**, i.e., the number of items in the range.
- We need $\Omega(s)$ time simply to report the items.
- Note that sometimes $s = 0$ and sometimes $s = n$; we therefore keep it as a separate parameter when analyzing the run-time.

Typical run-time: $O(\log n + s)$.

Range searches in existing dictionary realizations

Unsorted list/array/hash table: Range search requires $\Omega(n)$ time:
We have to check for each item explicitly whether it is in the range.

Sorted array: Range search in A can be done in $O(\log n + s)$ time:

range-search((18,45])

5	10	11	17	19	33	45	51	55	59
			$\uparrow i$			$\uparrow i'$			

- Using binary search, find i such that x is at (or would be at) $A[i]$.
- Using binary search, find i' such that x' is at (or would be at) $A[i']$
- Report all items $A[i+1 \ldots i'-1]$
- Report $A[i]$ and $A[i']$ if they are in range

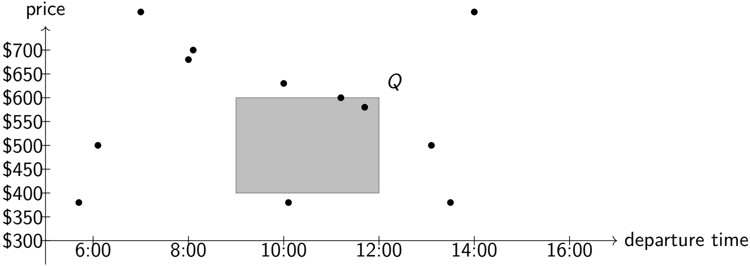
BST: Range searches can similarly be done in time $O(\text{height}+s)$ time.
We will see this in detail later.

Outline

Multi-Dimensional Data

Range searches are of special interest for **multi-dimensional data**.

Example: flights that leave between 9am and noon, and cost \$400-\$600



- Each item has d **aspects** (coordinates): $(x_0, x_1, \cdots, x_{d-1})$ so corresponds to a point in d-dimensional space
- We concentrate on $d = 2$, i.e., points in Euclidean plane
- (Orthogonal) ***d*-dimensional range search**: Given a **query rectangle** $Q = [x_1, x_1'] \times \cdots \times [x_d, x_d']$, find all points that lie within Q.

Multi-dimensional Range Search

The time for range searches depends on how the points are stored.

- Could store a 1-dimensional dictionary (where the key is some combination of the aspects.)
  Problem: Range search on one aspect is not straightforward
- Could use one dictionary for each aspect
  Problem: inefficient, wastes space
- **Better idea**: Design new data structures specifically for points.
  - Quadtrees
  - kd-trees
  - range-trees
- **Assumption**: Points are in **general position**:
  - No two points on a horizontal line.
  - No two points on a vertical line.

  This simplifies presentation; data structures can be generalized.

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Quadtrees

We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane.

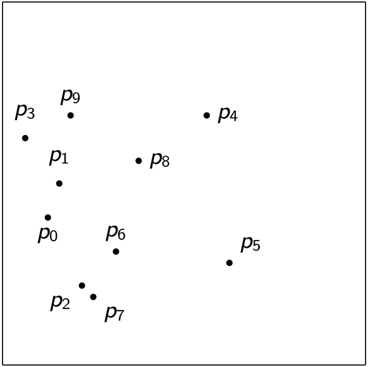
Find a **bounding box** $R = [0, 2^k) \times [0, 2^k)$: a square containing all points.

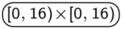
- Assume (after translation) that all coordinates are non-negative.
- Find max-coordinate in P, use the smallest k such that it is $< 2^k$.

Structure (and also how to *build* the quadtree that stores P):

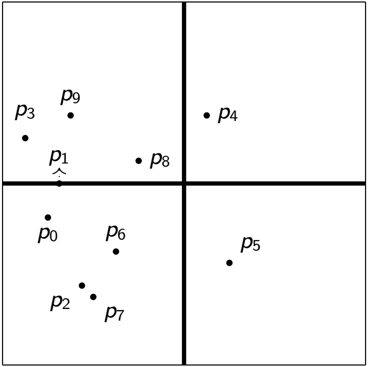
- Root r of the quadtree is associated with region R
- If R contains 0 or 1 points, then root r is a leaf that stores point.
- Else *split*: Partition R into four equal subsquares (**quadrants**) $R_{NE}, R_{NW}, R_{SW}, R_{SE}$
- Partition P into sets $P_{NE}, P_{NW}, P_{SW}, P_{SE}$ of points in these regions.
  - **Convention:** Points on split lines belong to right/top side
- Recursively build tree T_i for points P_i in region R_i and make them children of the root.

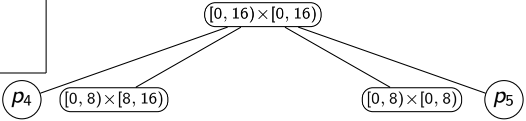
Quadtree example





Quadtree example





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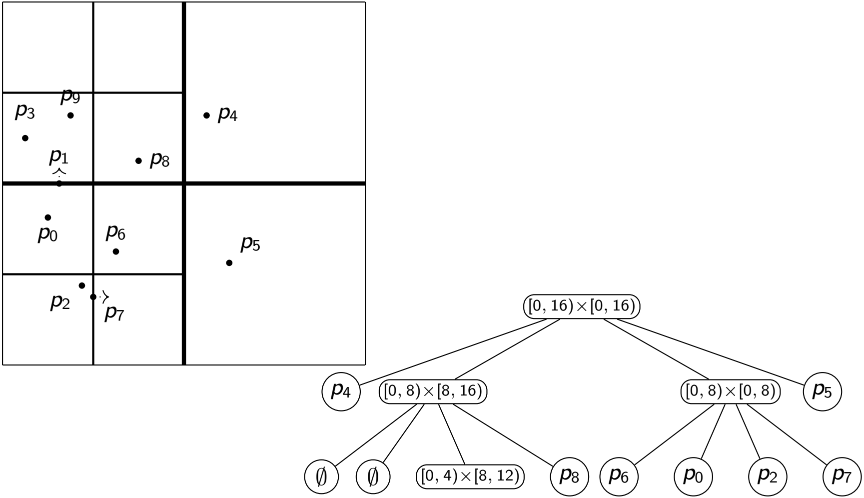

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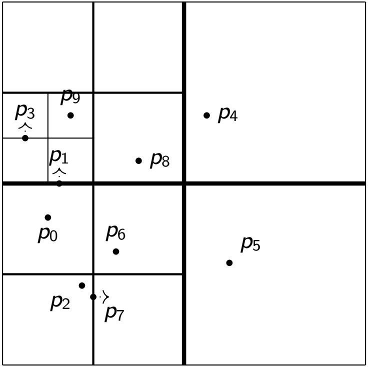
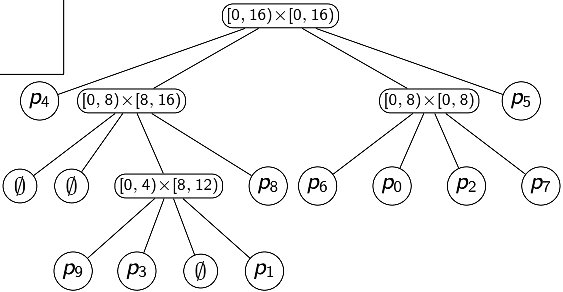
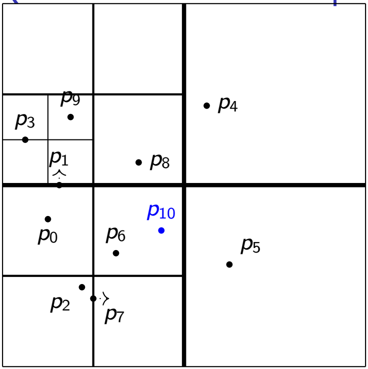




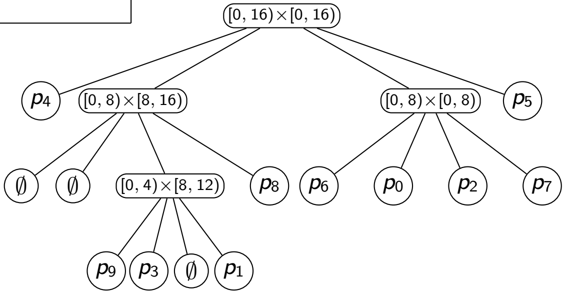
Quadtree Dictionary Operations

- *search*: Analogous to binary search trees and tries
- *insert*:
  - Search for the point
  - Split the leaf while there are two points in one region
- *delete*:
  - Search for the point
  - Remove the point
  - If its parent has only one point left: delete parent
    (and recursively all ancestors that have only one point left)

Quadtree Insert example



insert(p_{10})



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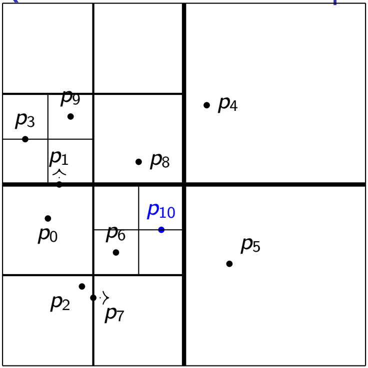

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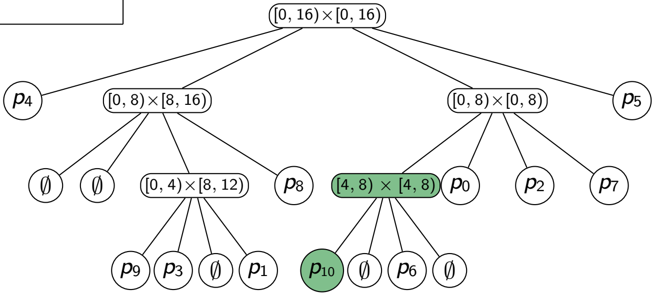


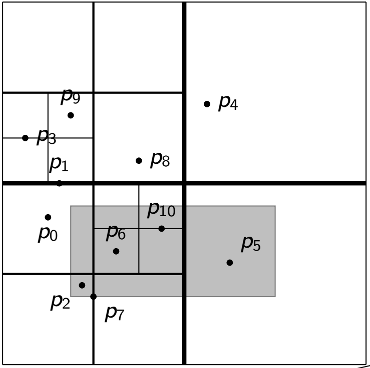
Quadtree Range Search

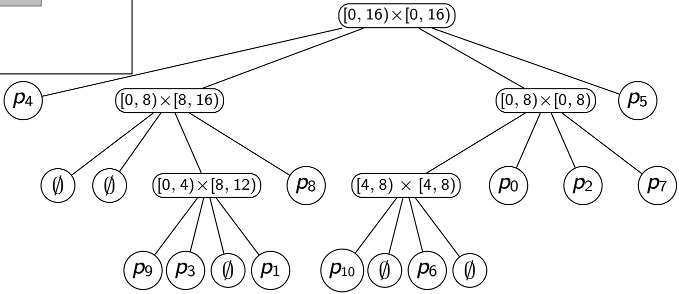
```
QTree::range-search(r ← root, Q)
r: The root of a quadtree, Q: Query-rectangle
    R ← region associated with node r
    if (R ⊆ Q) then                              // inside node, stop searching
            report all points below r and return
    else if (R ∩ Q is empty) then return         // outside node, stop searching

                                                 // boundary node, recurse
    if (r is a leaf) then
            p ← point stored at r
            if p is not NULL and in Q then report it and return
            else return
    for each child v of r do QTree::range-search(v, Q)
```

Note: We assume here that each node of the quadtree stores the associated square. Alternatively, these could be re-computed during the search (space-time tradeoff).

Quadtree range search example





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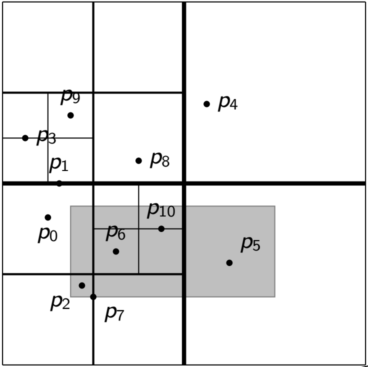
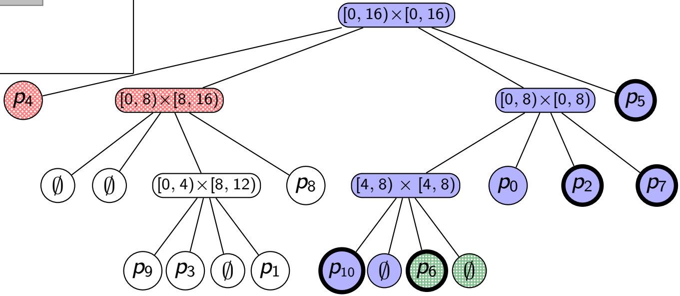


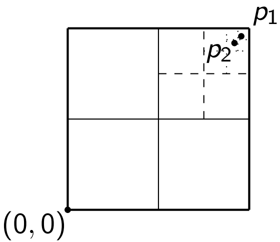
- Green: Search stopped due to $R \subseteq Q$.
- Red: Search stopped due to $R \cap Q = \emptyset$.
- Blue: Must continue search in children / evaluate.



Quadtree Analysis

Crucial for analysis: what is the height of a quadtree?

- Can have very large height for bad distributions of points.
- Even with $n = 3$ points, the height could be arbitrarily large.



- There exists a (weaker) bound that depends on the **spread factor** of points P:

$$\frac{\text{sidelength of } R}{\text{minimum distance between points in } P}$$

- Can show: height h of quadtree is in $\Theta(\log(\text{spread factor}))$
- Complexity to build initial tree: $\Theta(nh)$ worst-case
- Complexity of range search: $\Theta(nh)$ worst-case even if the answer is $\emptyset$

Quadtrees in other dimensions

- Quad-tree of 1-dimensional points:

“Points:” 0 9 12 14 24 26 28

Quadtrees in other dimensions

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"Points:"	0	9	12	14	24	26	28
(in base-2)	00000	01001	01100	01110	11000	11010	11100

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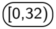

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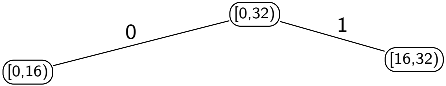

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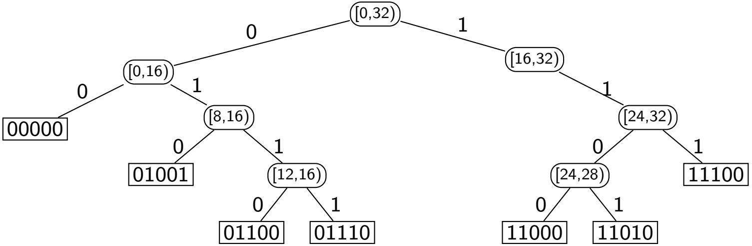

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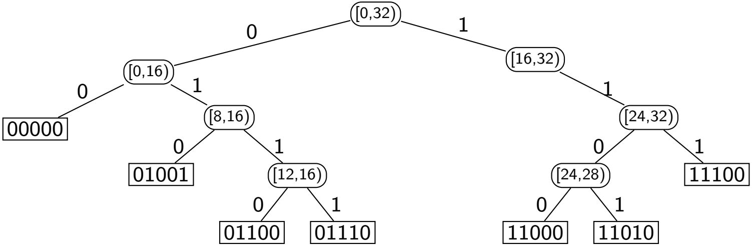
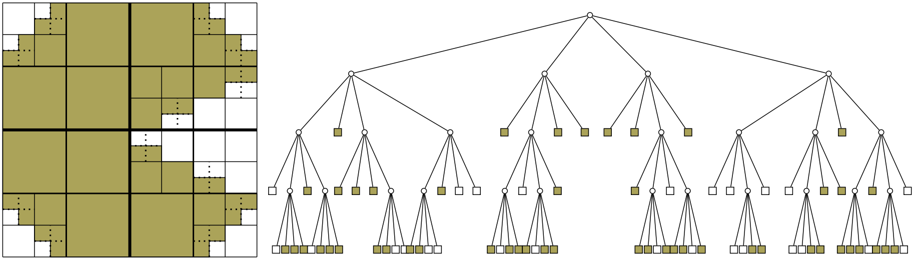


Same as a pruned trie

- Quadtrees also easily generalize to higher dimensions (split into octants $\rightarrow$ octrees, etc.) but are rarely used beyond dimension 3.

Quadtree summary

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (bit-shift!) if the width/height of bounding box R is a power of 2
- Space potentially wasteful, but good if points are well-distributed
- Variation: We could stop splitting earlier and allow up to K points in a leaf (for some fixed bound K).
- Variation: Use quad-tree to store pixelated images.



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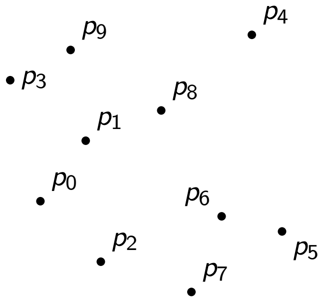
kd-trees

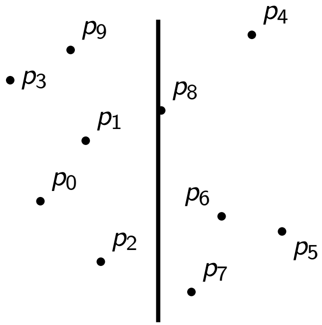
- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$
- Quadtrees split square into quadrants regardless of where points are
- (Point-based) kd-tree idea: Split the region at upper median of coordinates ($\rightsquigarrow$ roughly half of the point are in each subtree)
- Each node of the kd-tree keeps track of a **splitting line** in one dimension (2D: either vertical or horizontal)
- **Convention:** Points on split lines belong to right/top side
- Continue splitting, switching between vertical and horizontal lines, until every point is in a separate region

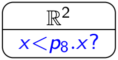
  (There are alternatives, e.g., split by the dimension that has better aspect ratios for the resulting regions.)

kd-tree example

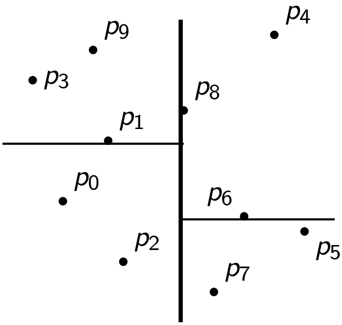


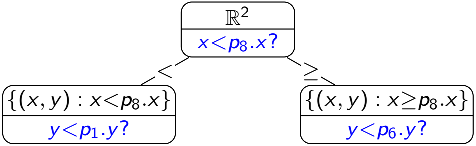
kd-tree example





kd-tree example





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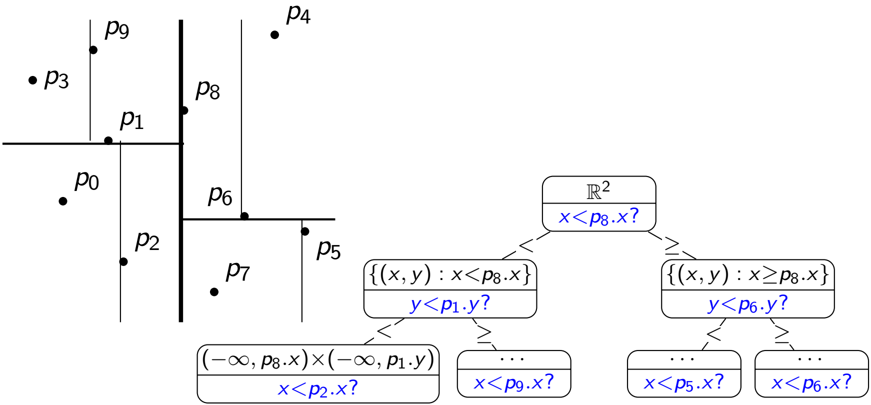
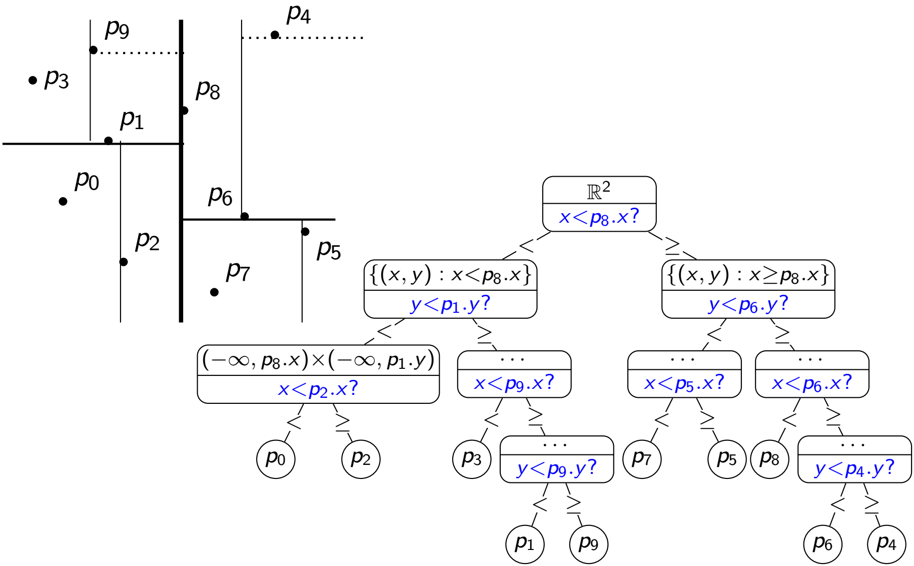


For ease of drawing, we will usually not show the associated regions.

kd-tree example



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Constructing kd-trees

Build kd-tree with initial split by x on points P:

- If $|P| \leq 1$ create a leaf and return.
- Else $X :=$ *randomized-quick-select*$(P, \lfloor \frac{n}{2} \rfloor)$ (select by x-coordinate)
- Partition P by x-coordinate into $P_{x<X}$ and $P_{x \geq X}$
  - $\lfloor \frac{n}{2} \rfloor$ points on one side and $\lceil \frac{n}{2} \rceil$ points on the other.
    (Recall: Points in general position.)
- Create left subtree recursively (splitting by y) for points $P_{x<X}$.
- Create right subtree recursively (splitting by y) for points $P_{x \geq X}$.

Building with initial y-split symmetric.

Constructing kd-trees

Run-time:

- Find X and partition P in $\Theta(n)$ expected time using *randomized-quick-select*.
- Both subtrees have $\approx n/2$ points.

$$T^{\exp}(n) = 2T^{\exp}(n/2) + O(n) \quad \text{(sloppy recurrence)}$$

  This resolves to $\Theta(n \log n)$ expected time.
- This can be reduced to $\Theta(n \log n)$ *worst-case* time by pre-sorting (no details).

Height: $h(1) = 0$, $h(n) \leq h(\lceil n/2 \rceil) + 1$.

- This resolves to $O(\log n)$ (specifically $\lceil \log n \rceil$).
- This is tight (binary tree with n leaves)

Space: All interior nodes have exactly two children.

- Therefore have $n - 1$ interior nodes.
- Space is $\Theta(n)$.

kd-tree Dictionary Operations

- *search* (for single point): as in binary search tree using indicated coordinate
- *insert*: search, insert as new leaf.
- *delete*: search, remove leaf.

Problem: After insert or delete, the split might no longer be at exact median and the height is no longer guaranteed to be $\lceil \log_2 n \rceil$.

We can maintain $O(\log n)$ height by occasionally re-building entire subtrees. (No details.) But *range-search* will be slower.

kd-trees do not handle insertion/deletion well.

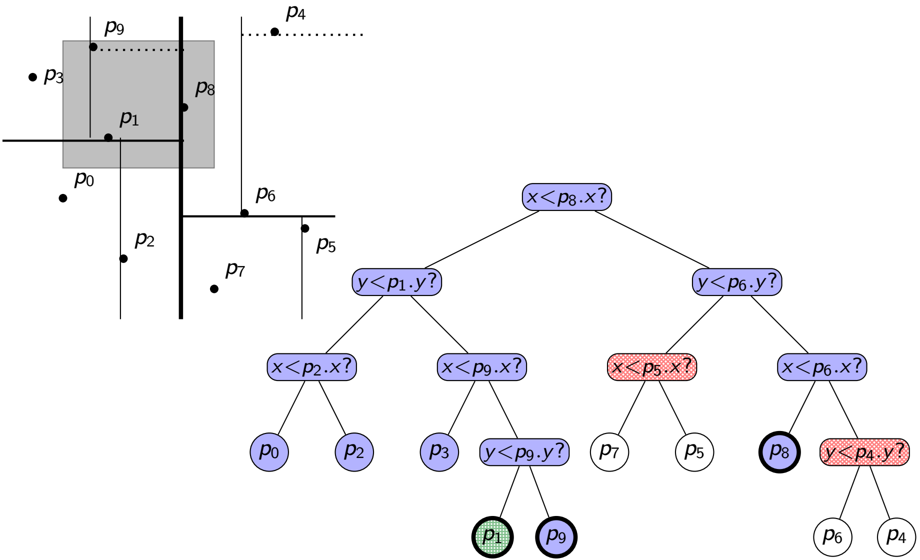
kd-tree Range Search

- Range search is *exactly* as for quad-trees, except that there are only two children and leaves always store points.

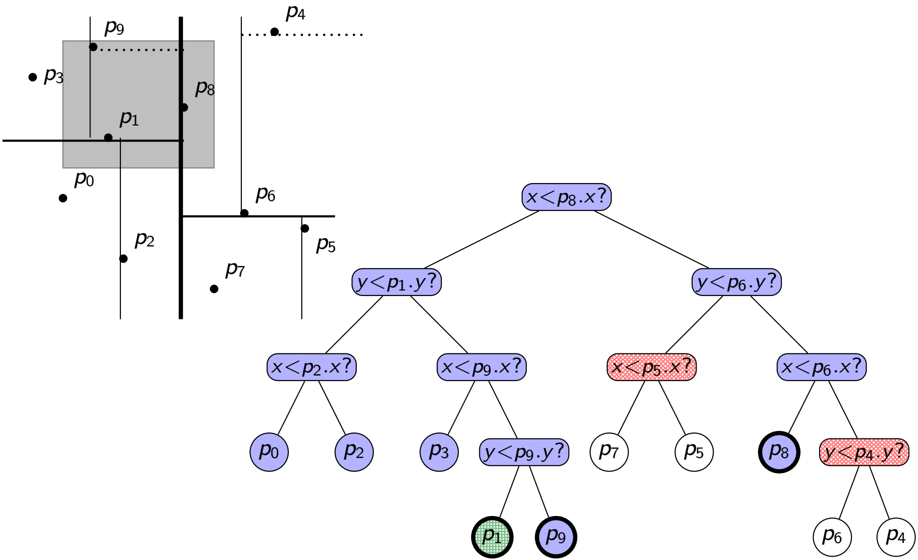
```
kdTree::range-search(r ← root, Q)
r: The root of a kd-tree, Q: Query-rectangle
    R ← region associated with node r
    if (R ⊆ Q) then report all points below r; return
    if (R ∩ Q is empty) then return
    if (r is a leaf) then
        p ← point stored at r
        if p is in Q return p
        else return
    for each child v of r do kdTree::range-search(v, Q)
```

- We assume again that each node stores its associated region.
- To save space, we could instead pass the region as a parameter and compute the region for each child using the splitting line.

kd-tree: Range Search Example



kd-tree: Range Search Example



Red: Search stopped due to $R \cap Q = \emptyset$. Green: Search stopped due to $R \subseteq Q$.

kd-tree: Range Search Complexity

- We spend $O(1)$ time at each visited node, except in line 2.
- All calls to line 2 together take $O(s)$ time (recall: s is the output-size)
- **Observe**: # visited nodes is $O(\beta(n))$
  where $\beta(n)$ is the number of "boundary" nodes (blue):
  - *kdTree::range-search* was called.
  - Neither $R \subseteq Q$ nor $R \cap Q = \emptyset$

- **Can show:** $\beta(n)$ satisfies the following recurrence relation:

$$\beta(n) \leq 2\beta(n/4) + O(1)$$

- This implies $\beta(n) \in O(\sqrt{n})$
- Therefore, the complexity of range search in kd-trees is $O(s + \sqrt{n})$

kd-tree: Higher Dimensions

- kd-trees for d-dimensional space:
  - At the root the point set is partitioned based on the first coordinate
  - At the subtrees of the root the partition is based on the second coordinate
  - At depth $d-1$ the partition is based on the last coordinate
  - At depth d we start all over again, partitioning on first coordinate
- **Storage**: $O(n)$
- **Height**: $O(\log n)$
- **Construction time**: $O(n \log n)$
- **Range search time**: $O(s + n^{1-1/d})$

This assumes that d is a constant.

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Towards Range Trees

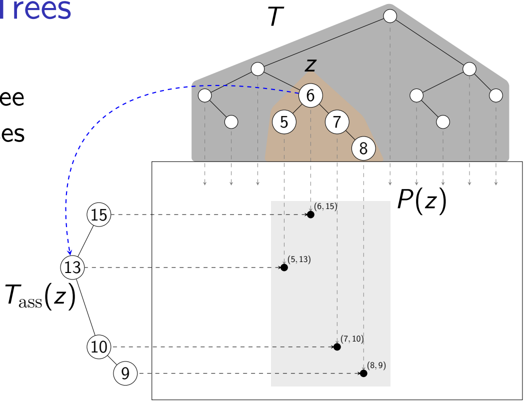
- Both Quadtrees and kd-trees are intuitive and simple.
- But: both may be very slow for range searches.
- Quadtrees are also potentially wasteful in space.

New idea: **Range trees**

- **Tree of trees** (a *multi-level* data structure)
  - So far, nodes in our trees stored a key-value pair and references to children and (maybe) the parent
  - But we can store much more in a node!
  - Here: Each node stores in another binary search tree (!)
- They are wasteful in space, but permit much faster range search.

2-dimensional Range Trees

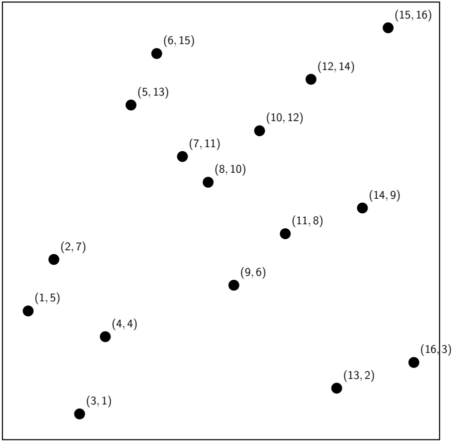
Primary structure:
Balanced binary search tree T that stores P and uses *x-coordinates* as keys.



Every node z of T stores an **associate structure** $T_{\mathrm{ass}}(z)$:

- Let $P(z)$ be all points in subtree of z in T (including point at z)
- $T_{\mathrm{ass}}(z)$ stores $P(z)$ in a balanced binary search tree, using the *y-coordinates* as key
- Note: Point of z is not necessarily the root of $T_{\mathrm{ass}}(z)$

Bigger example



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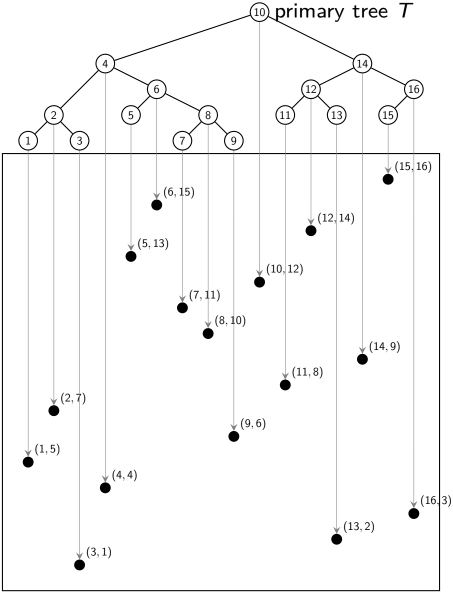
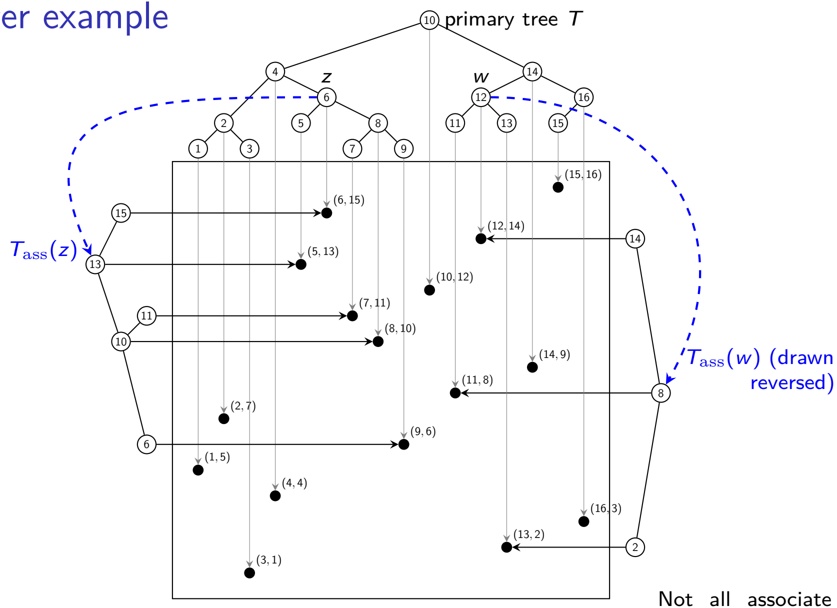


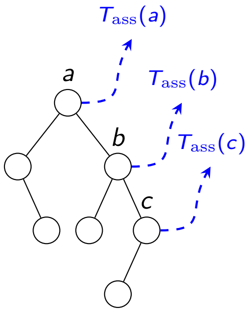
Bigger example



Not all associate trees are shown.

Range Tree Space Analysis

- Primary tree T uses $O(n)$ space.
- How many nodes do all associate trees together have?



  - point of a is only in associate tree $T_{\text{ass}}(a)$
  - point of b is in associate trees $T_{\text{ass}}(a)$, $T_{\text{ass}}(b)$
  - point of c is in associate trees $T_{\text{ass}}(a)$, $T_{\text{ass}}(b)$, $T_{\text{ass}}(c)$
  - **Key insight**: point of z is in associate tree $T_{\text{ass}}(u)$ if and only if u is an ancestor of z in T
  - So every point belongs to $O(\log n)$ associate trees.
  - So all associate trees together use $O(n \log n)$ space.
- A range-tree with n points uses $O(n \log n)$ space.

This is tight for some primary trees.

Range Trees Operations

- *search*: search by x-coordinate in T
- *insert*: First, insert point by x-coordinate into T.
  Then, walk back up to the root and insert the point by y-coordinate in *all* associate trees $T_{\text{ass}}(z)$ of nodes z on path to the root.
- *delete*: analogous to insertion
- **Problem:** We want the binary search trees to be balanced.
  - This makes *insert*/*delete* very slow if we use AVL-trees.
    (A rotation at v changes $P(v)$ and hence requires a re-build of $T_{\text{ass}}(v)$.)
  - **Solution**: Completely rebuild highly unbalanced subtrees (no details)
  - Run-time for *insert*/*delete* becomes $O(\log^2 n)$ amortized.

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  - **Solution**: Completely rebuild highly unbalanced subtrees (no details)
  - Run-time for *insert*/*delete* becomes $O(\log^2 n)$ amortized.
- *range-search*: search by x-range in T.
  Among found points, search by y-range in some associated trees.
- Must understand first: How to do (1-dimensional) range search in binary search tree?

BST Range Search recursive

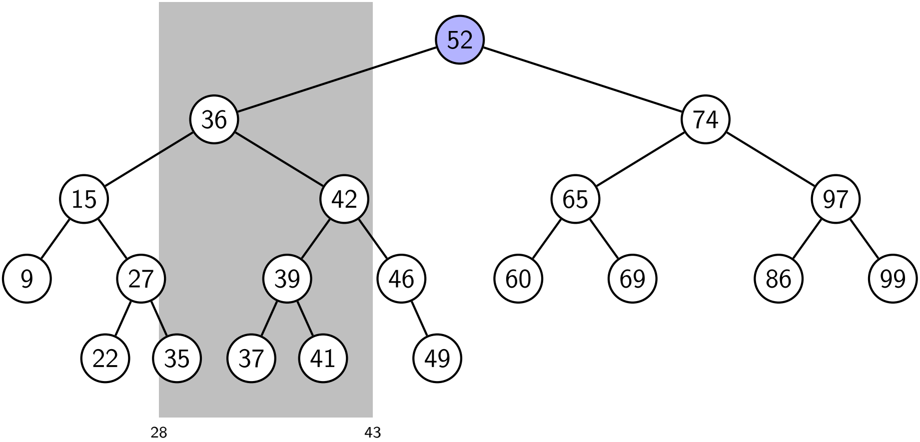
BST::range-search-recursive$(r \leftarrow root, x_1, x_2)$
r: root of a binary search tree, x_1, x_2: search keys
Returns keys in subtree at r that are in range $[x_1, x_2]$

```
if r = NULL then return
if x1 ≤ r.key ≤ x2 then
      L ← BST::range-search-recursive(r.left, x1, x2)
      R ← BST::range-search-recursive(r.right, x1, x2)
      return L ∪ r.{key} ∪ R
if r.key < x1 then
      return BST::range-search-recursive(r.right, x1, x2)
if r.key > x2 then
      return BST::range-search-recursive(r.left, x1, x2)
```

Keys are reported in in-order, i. e., in sorted order.

BST Range Search example

BST::range-search-recursive$(T, 28, 43)$



BST Range Search example

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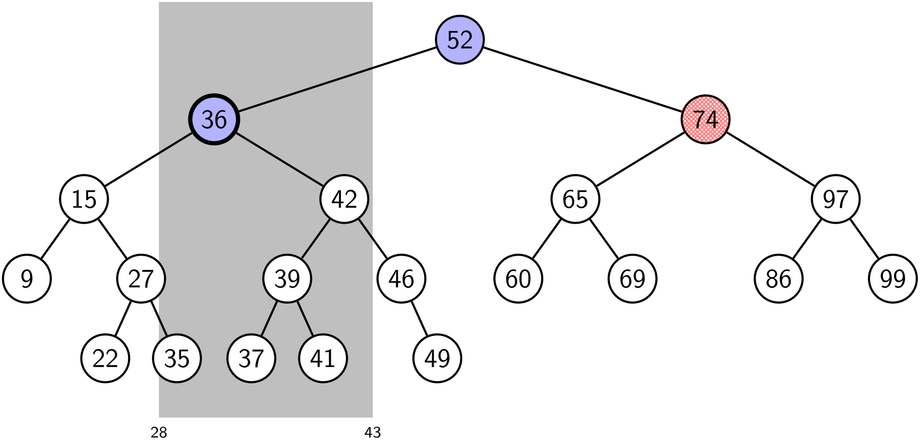

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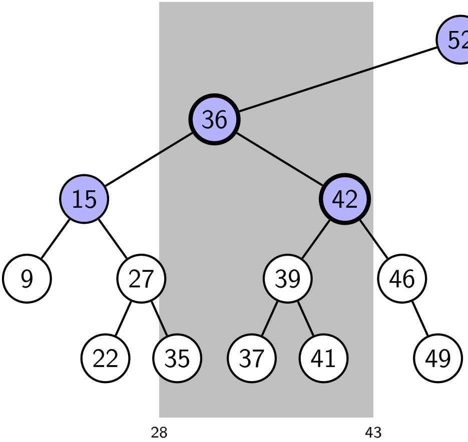

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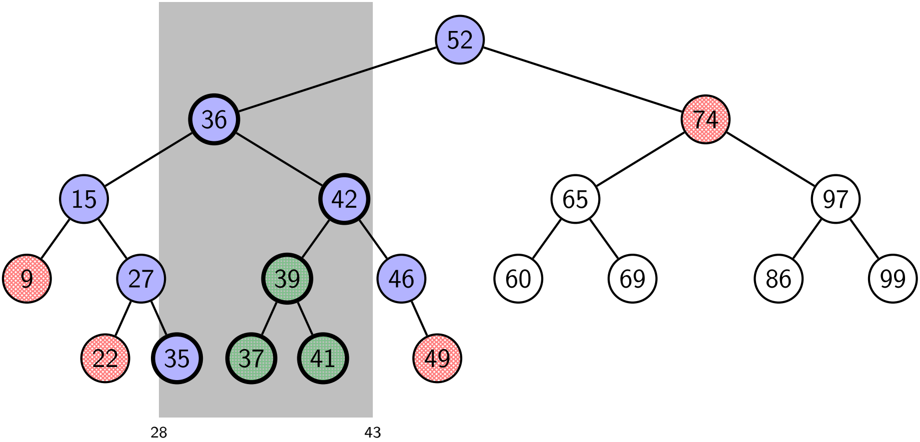

BST Range Search example

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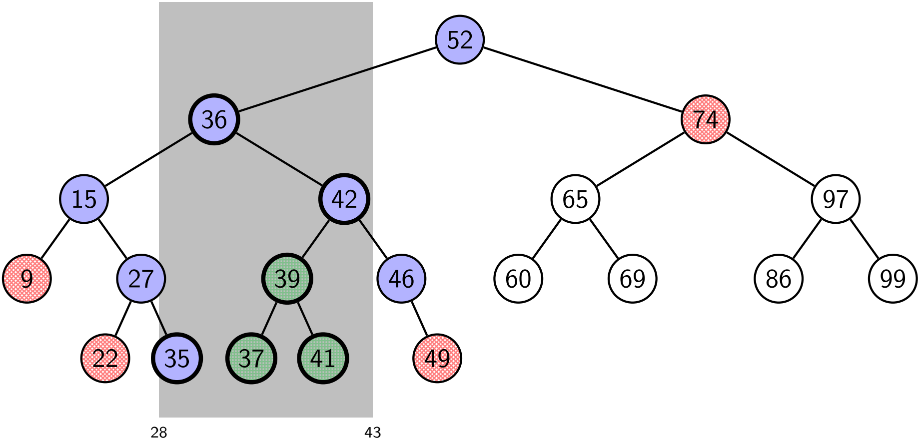
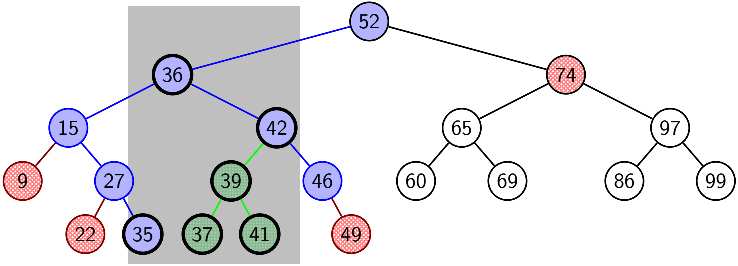


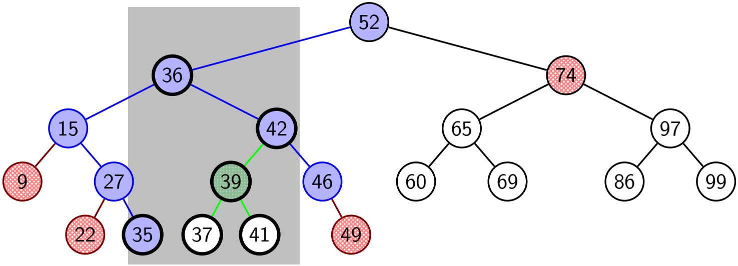
Note: Search from 39 was unnecessary: *all* its descendants are in range.

BST Range Search re-phrased



- Search for left boundary x_1: this gives path P_1
- Search for right boundary x_2: this gives path P_2
- This partitions T into three groups: outside, on, or between the paths.
- This classification will be crucial later!

BST Range Search re-phrased

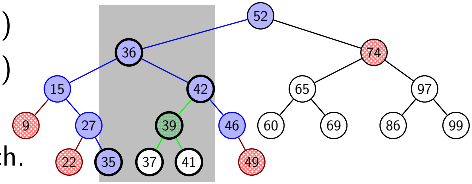


- boundary nodes: nodes in P_1 or P_2
  - For each boundary node, test whether it is in the range.
- outside nodes: nodes that are left of P_1 or right of P_2
  - These are *not* in the range, we do not visit them.
- inside nodes: nodes that are right of P_1 and left of P_2
  - We keep a list of the topmost inside nodes.
  - All descendants of such a node are *in* the range.
    For a 1d range search, report them.

BST Range Search analysis

Assume that the binary search tree is balanced:

- Search for path P_1: $O(\log n)$
- Search for path P_2: $O(\log n)$
- $O(\log n)$ boundary nodes
- We spend $O(1)$ time on each.



BST Range Search analysis

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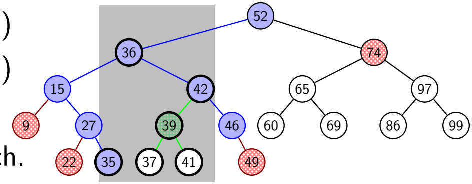


- We spend $O(1)$ time per topmost inside node v.
  - They are children of boundary nodes, so this takes $O(\log n)$ time.
- For 1d range search, also report the descendants of v.
  - We have $\sum_{z \text{ topmost inside}} \#\{\text{descendants of } z\} \leq s$ since subtrees of topmost inside nodes are disjoint. So this takes time $O(s)$ overall.

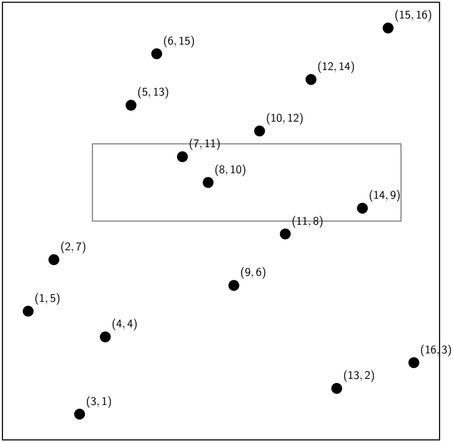
Run-time for 1d range search: $O(\log n + s)$. This is no faster overall, but topmost inside nodes will be important for 2d range search.

Range Trees: Range Search

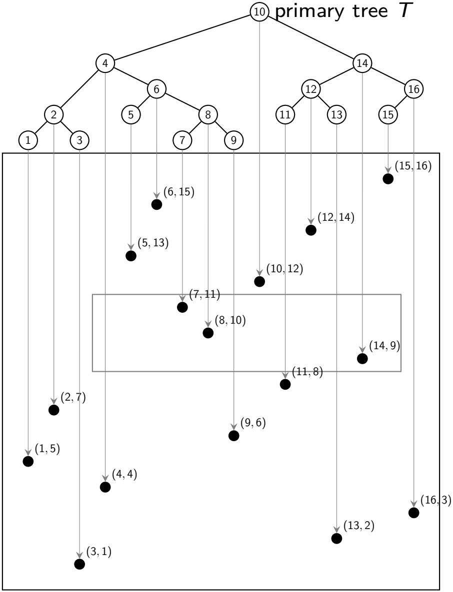
Range search for $Q = [x_1, x_2] \times [y_1, y_2]$ is a two stage process:

- Perform a range search (on the x-coordinates) for the interval $[x_1, x_2]$ in primary tree T (*BST::range-search*(T, x_1, x_2))
- Get boundary and topmost inside nodes as before.
- For every boundary node, test to see if the corresponding point is within the region Q.
- For every topmost inside node v:
  - Let $P(z)$ be the points in the subtree of z in T.
  - We know that all x-coordinates of points in $P(z)$ are within range.
  - Recall: $P(z)$ is stored in $T_{\text{ass}}(z)$.
  - To find points in $P(z)$ where the y-cordinates are within range as well, perform a range search in $T_{\text{ass}}(z)$: *BST::range-search*$(T_{\text{ass}}(z), y_1, y_2)$

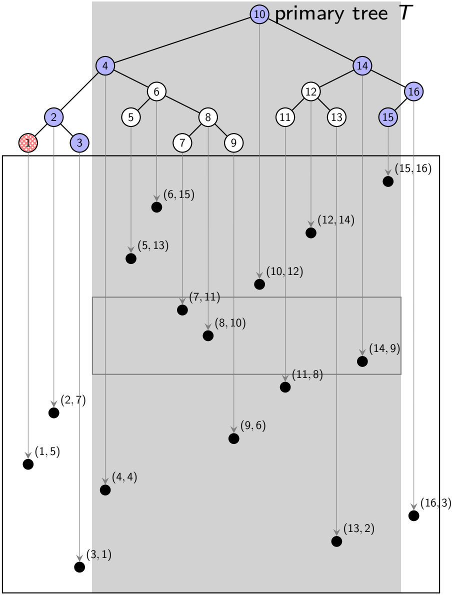
Range tree range search example



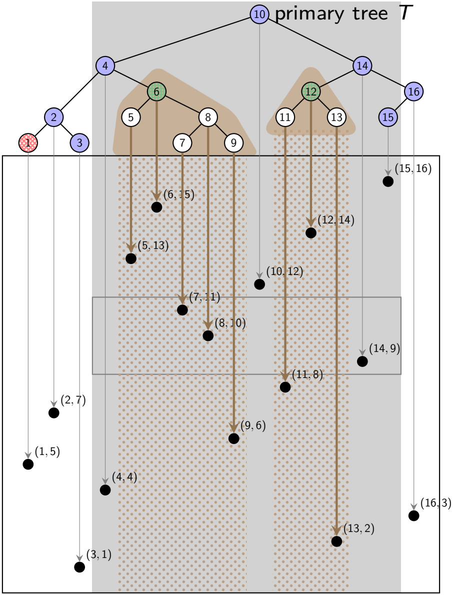
Range tree range search example



Range tree range search example



Range tree range search example



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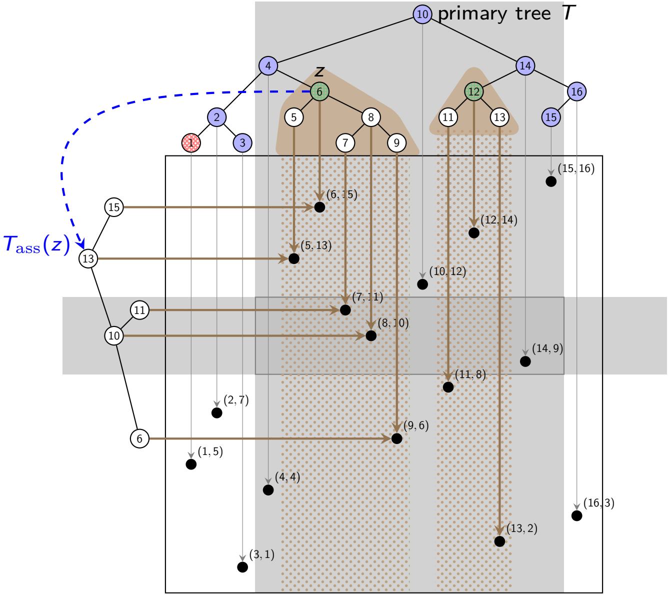
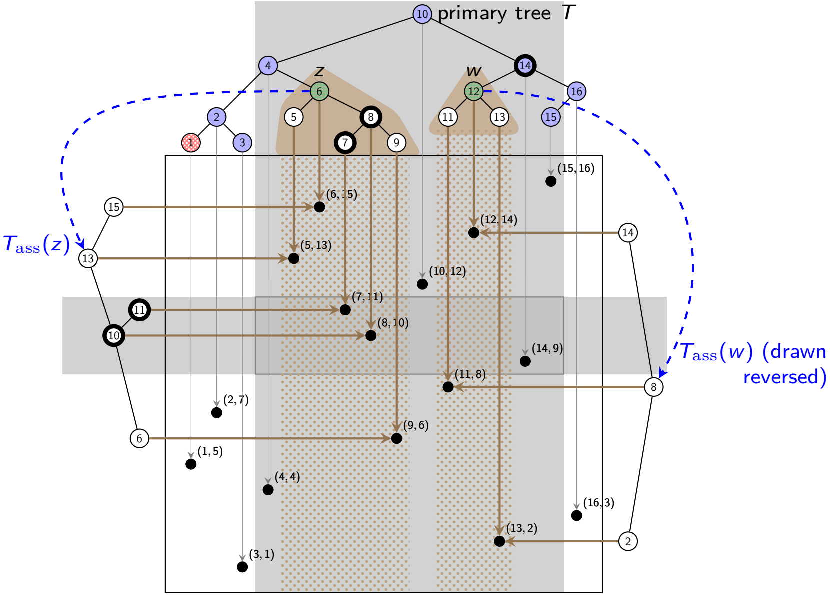


Range tree range search example



Range Trees: Range Search Run-time

- $O(\log n)$ time to find boundary and topmost inside nodes in primary tree.
- There are $O(\log n)$ such nodes.
- $O(\log n + s_v)$ time for each topmost inside node v, where s_v is the number of points in $T_{\mathrm{ass}}(v)$ that are reported
- Two topmost inside nodes have no common point in their trees
  $\Rightarrow$ every point is reported in at most one associate structure
  $\Rightarrow \sum_{v \text{ topmost inside}} s_v \leq s$

Time for range search in range-tree is proportional to

$$\sum_{v \text{ topmost inside}} (\log n + s_v) \in O(\log^2 n + s)$$

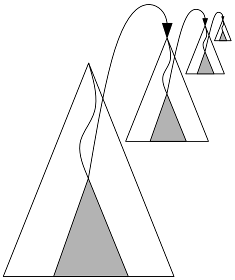
(There are ways to make this even faster. No details.)

Range Trees: Higher Dimensions

- Range trees can be generalized to d-dimensional space.

Space	$O(n\,(\log n)^{d-1})$
Construction time	$O(n\,(\log n)^{d})$
Range search time	$O(s + (\log n)^{d})$

(Note: d is considered to be a constant.)



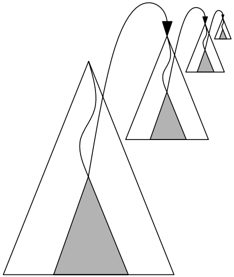
Range Trees: Higher Dimensions

- Range trees can be generalized to d-dimensional space.

Space	$O(n\,(\log n)^{d-1})$	kd-trees: $O(n)$
Construction time	$O(n\,(\log n)^{d})$	kd-trees: $O(n \log n)$
Range search time	$O(s + (\log n)^{d})$	kd-trees: $O(s + n^{1-1/d})$

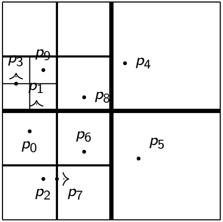
(Note: d is considered to be a constant.)

- Space/time trade-off compared to kd-trees.

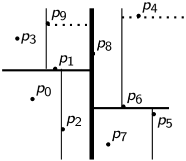


Outline

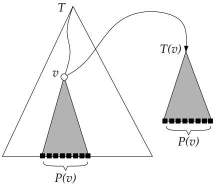
Range search data structures summary



- Quadtrees
  - simple (also for dynamic set of points)
  - work well only if points evenly distributed
  - wastes space for higher dimensions



- kd-trees
  - linear space
  - range search time $O(\sqrt{n} + s)$
  - inserts/deletes destroy balance and range search time (no simple fix)



- range-trees
  - range search time $O(\log^2 n + s)$
  - wastes some space
  - inserts/deletes destroy balance (can fix this with occasional rebuilt)

Convention: Points on split lines belong to right/top side.