CS 240 – Data Structures and Data Management

Module 4: Dictionaries

Mark Petrick, Éric Schost

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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Outline

4 Dictionaries and Balanced Search Trees

- ADT Dictionary
- Binary Search Trees
- AVL Trees
- Insertion in AVL Trees
- Restructuring a BST: Rotations
- AVL insertion revisited
- Deletion in AVL Trees

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ADT Dictionary (review)

Dictionary: An ADT consisting of a collection of items, each of which contains

- a *key*
- some *data* (the "value")

and is called a *key-value pair* (KVP). Keys can be compared and are (typically) unique.

Operations:

- *search*(k) (also called *lookup*(k))
- *insert*(k, v)
- *delete*(k) (also called *remove*(k)))
- optional: *successor*, *join*, *is-empty*, *size*, *etc.*

Examples: symbol table, license plate database

Elementary Realizations (review)

Common assumptions:

- Dictionary has n KVPs
- Each KVP uses constant space
  (if not, the "value" could be a pointer)
- Keys can be compared in constant time

Unordered array or linked list

search $\Theta(n)$

insert $\Theta(1)$ (except array occasionally needs to resize)

delete $\Theta(n)$ (need to search)

Ordered array

search $\Theta(\log n)$ (via binary search)

insert $\Theta(n)$

delete $\Theta(n)$

Binary Search (review)

Only applies to a *sorted array*:

0	1	2	3	4	5	6
30	40	70	90	100	120	140

```
*binary-search*$(A, n, k)$
$A$: Sorted array of size $n$, $k$: key
$\ell \leftarrow 0$, $r \leftarrow n-1$
**while** $(\ell \leq r)$
    $m \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$
    **if** $(A[m]$ equals $k)$ **then return** "found at $A[m]$"
    **else if** $(A[m] < k)$ **then** $\ell \leftarrow m+1$
    **else** $r \leftarrow m-1$
**return** "not found, but would be between $A[\ell-1]$ and $A[\ell]$"
```

We will return to binary search (and sometimes improve it!) later.

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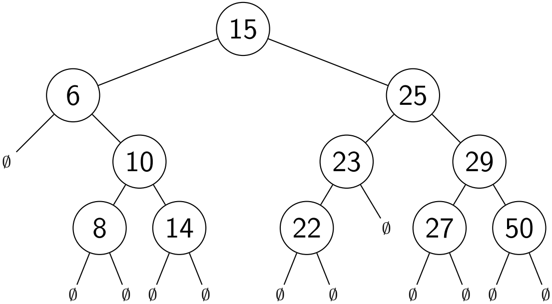
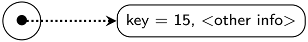
Binary Search Trees (review)

Structure	Binary tree: all nodes have two (possibly empty) subtrees Every node stores a KVP Empty subtrees usually not shown
Ordering	Every key k in $T.left$ is less than the root key. Every key k in $T.right$ is greater than the root key.



(In our examples we only show the keys, and we show them directly in the node. A more accurate picture would be

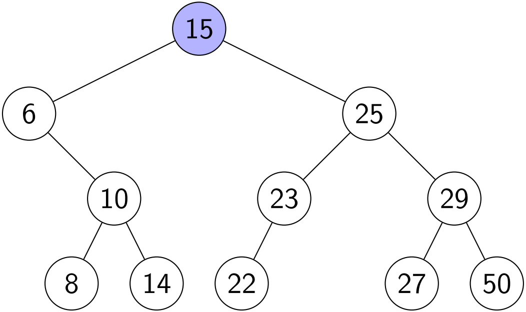


)

BST as realization of ADT Dictionary (review)

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

Example: *BST::search*(24)



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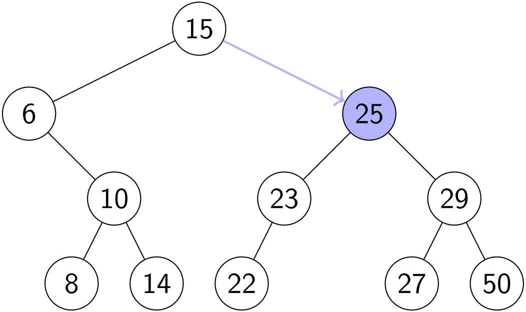

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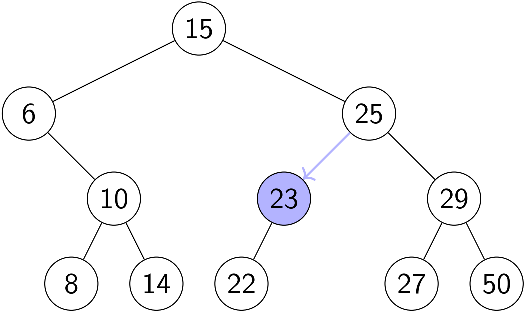

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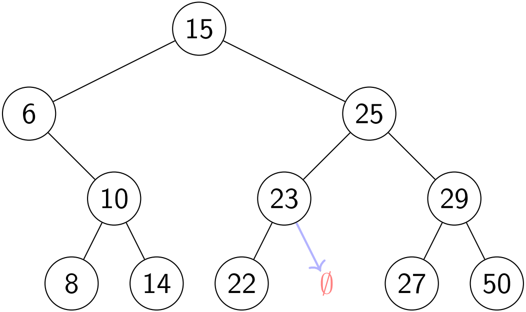
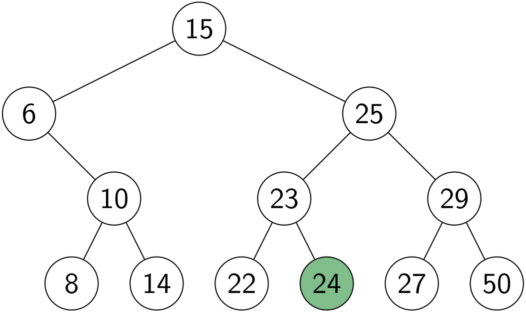


BST as realization of ADT Dictionary (review)

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

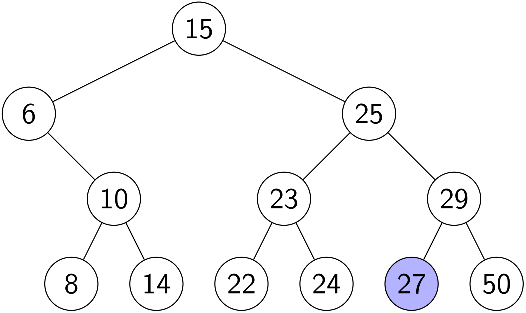
BST::insert(k, v) Search for k, then insert (k, v) as new node

Example: *BST::insert*$(24, v)$



Deletion in a BST

- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.



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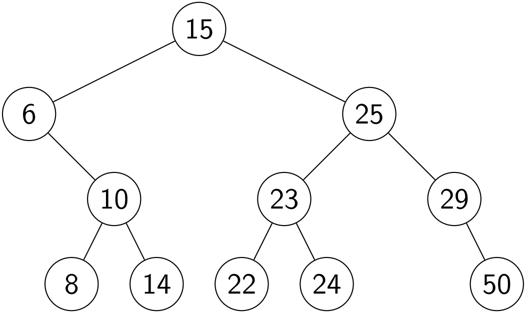

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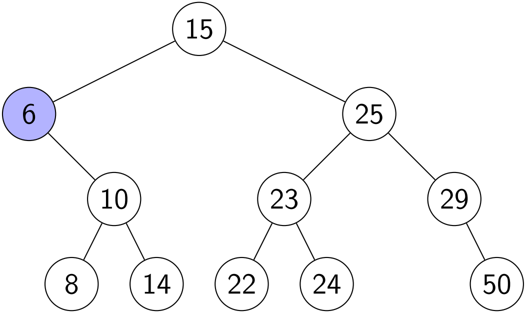

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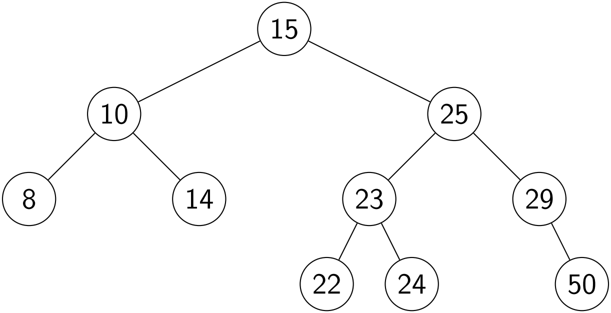

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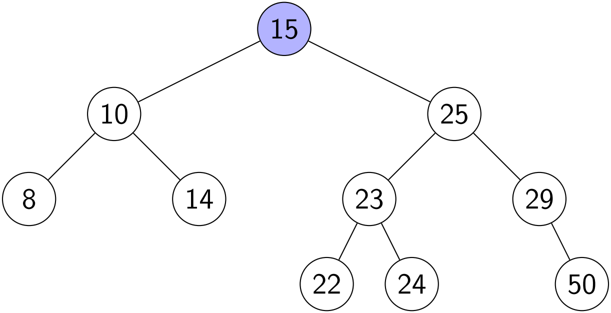
(Successor: next-smallest among all keys in the dictionary.)



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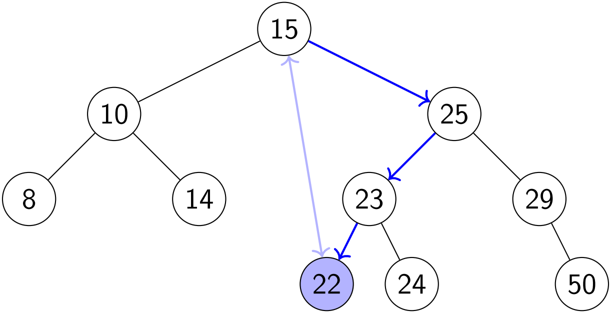

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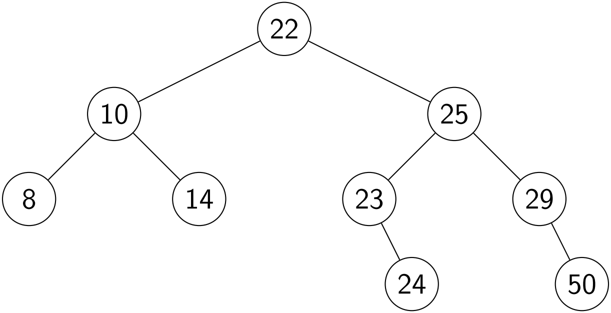

Height of a BST

BST::search, *BST::insert*, *BST::delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are inserted one-at-a-time, how big is h?

- Worst-case: $n - 1 = \Theta(n)$
- Best-case: $\Theta(\log n)$.
  Any binary tree with n nodes has height $h \geq \log(n+1) - 1$
  (Layer i has at most 2^i nodes. So $n \leq \sum_{i=0}^{h} 2^i = 2^{h+1} - 1$).

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Goal: Create subclasses of BSTs where the height is *always* good.

- Impose a structural property.
- Argue that the property implies logarithmic height.
- Discuss how to maintain the property during operatons.

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AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962, an **AVL Tree** is a BST with an additional **height-balance property** at every node:

The heights of the left and right subtree differ by at most 1.

Rephrase: If node v has left subtree L and right subtree R, then

$$\textbf{balance}(v) := \textit{height}(R) - \textit{height}(L) \text{ must be in } \{-1, 0, 1\}$$

$\textit{balance}(v) = -1$ means v is *left-heavy*

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$\mathit{balance}(v) = -1$ means v is *left-heavy*

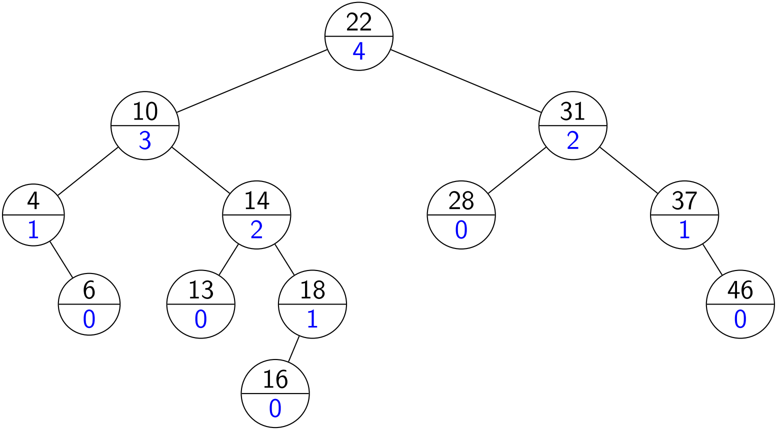
$\mathit{balance}(v) = +1$ means v is *right-heavy*

- Need to store at each node v the height of the subtree rooted at it

(There are ways to implement AVL-trees where we only store $\mathit{balance}(v)$, so fewer bits. But the code gets more complicated (no details).)

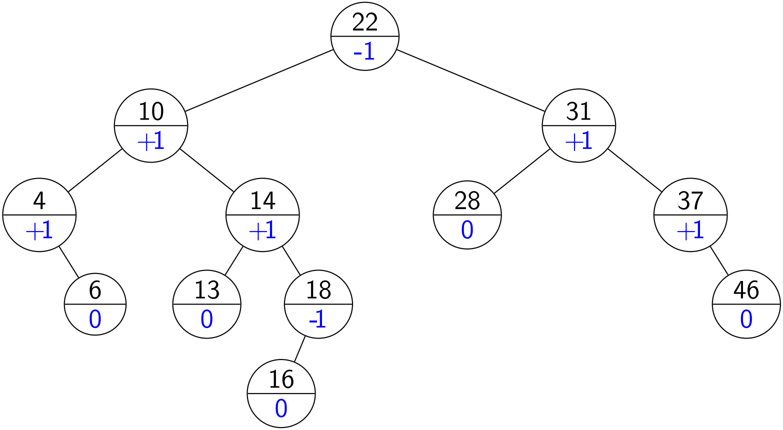
AVL tree example

(The lower numbers indicate the height of the subtree.)



AVL tree example

Alternative: store balance (instead of height) at each node.



Height of an AVL tree

Theorem: An AVL tree on n nodes has $\Theta(\log n)$ height.
$\Rightarrow$ *search*, *BST::insert*, *BST::delete* all cost $\Theta(\log n)$ in the *worst case!*

Proof:

- Define $N(h)$ to be the *least* number of nodes in a height-h AVL tree.
- What is a recurrence relation for $N(h)$?
- What does this recurrence relation resolve to?

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AVL insertion

To perform *AVL::insert*(k, v):

- First, insert (k, v) with the usual BST insertion.
- We assume that this returns the new leaf z where the key was stored.
- Then, move up the tree from z.

  (We assume for this that we have parent-links. This can be avoided if *BST::insert* returns the full path to z.)

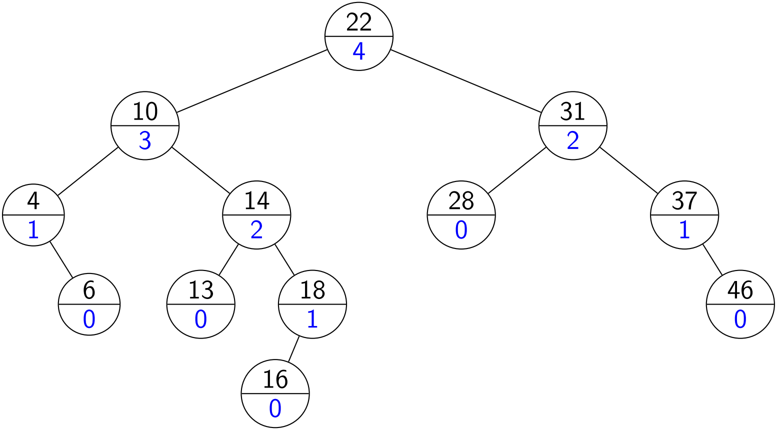
- Update height (easy to do in constant time):

```
setHeightFromSubtrees(u)
1.    $u.height \leftarrow 1 + \max\{u.left.height, u.right.height\}$
```

- If the height difference becomes ±2 at node z, then z is **unbalanced**. Must re-structure the tree to rebalance.

AVL Insertion Example

Example: *AVL::insert*(8)



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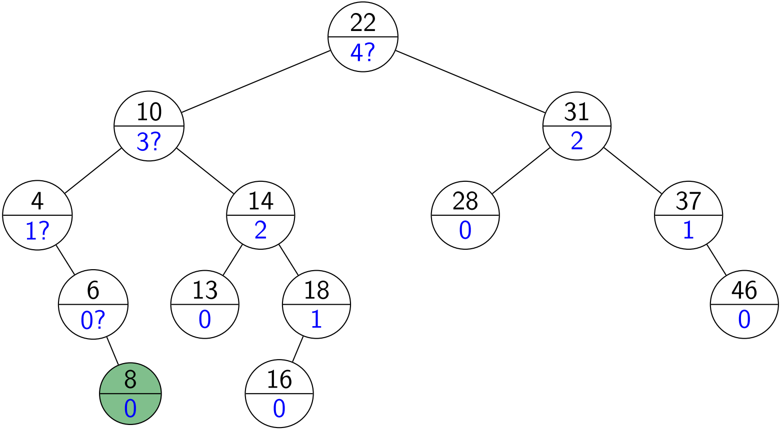

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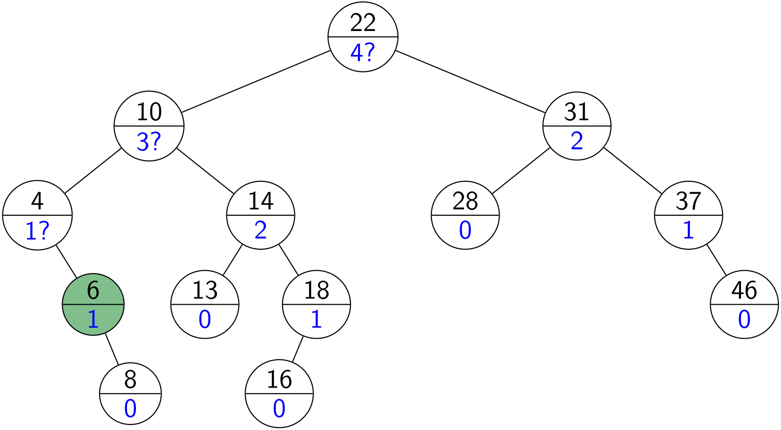

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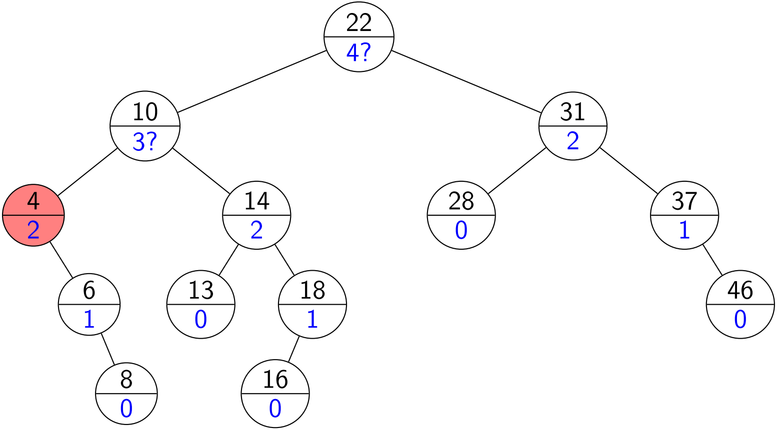

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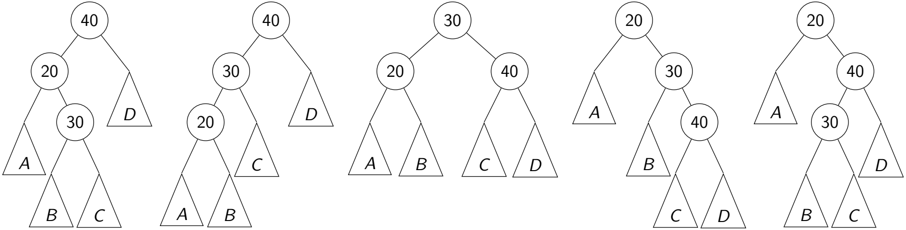
Changing structure without changing order

Note: There are many different BSTs with the same keys.

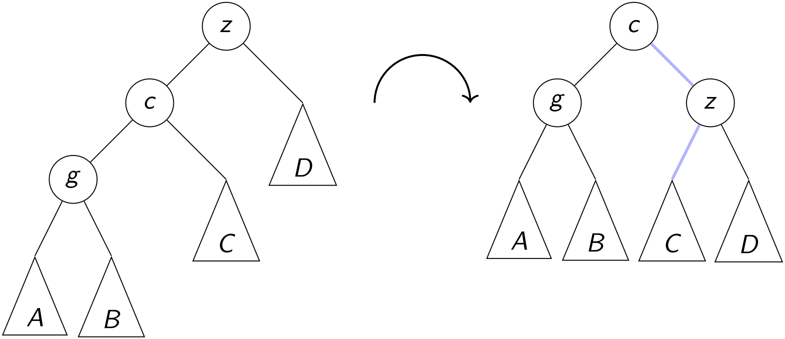


Goal: Change the *structure* locally nodes without changing the *order*.

Longterm goal: Restructure such the subtree becomes balanced.

Right Rotation

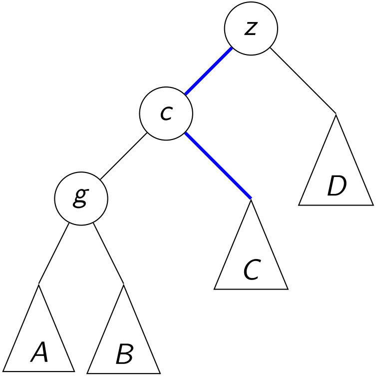
This is a **right rotation** on node *z*:



```
rotate-right(z)
1.  $c \leftarrow z.left$, $z.left \overset{p}{\leftarrow} c.right$, $c.right \overset{p}{\leftarrow} z$
2.  setHeightFromSubtrees(z), setHeightFromSubtrees(c)
3.  return c          // returns new root of subtree
```

(Notation $\overset{p}{\leftarrow}$ means ‘also change parent-reference of right-hand-side’)

Why do we call this a rotation?



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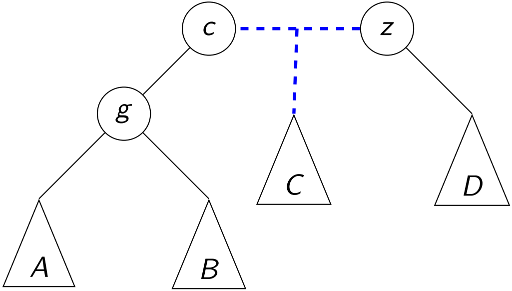

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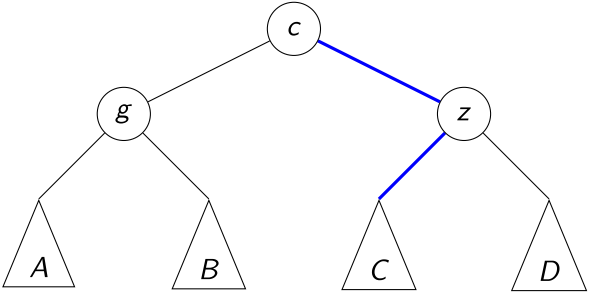
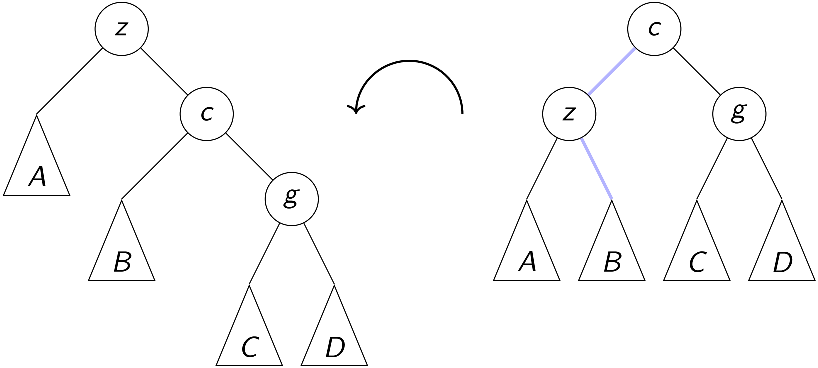


Left Rotation

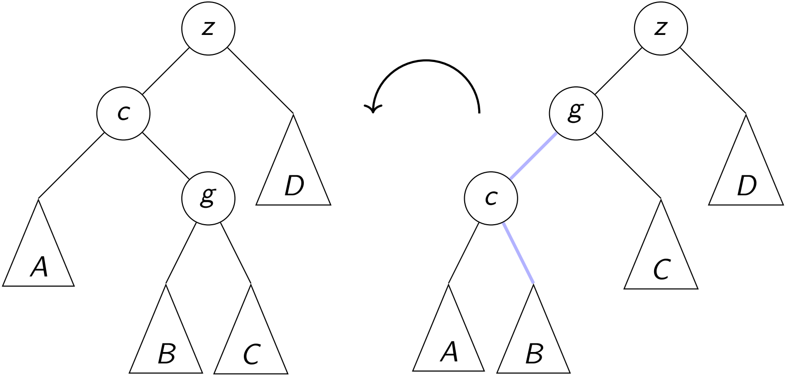
Symmetrically, this is a **left rotation** on node z:



Again, only two links need to be changed and two heights updated.
Useful to fix right-right imbalance.

Double Right Rotation

This is a **double right rotation** on node z:



First, a left rotation at c.

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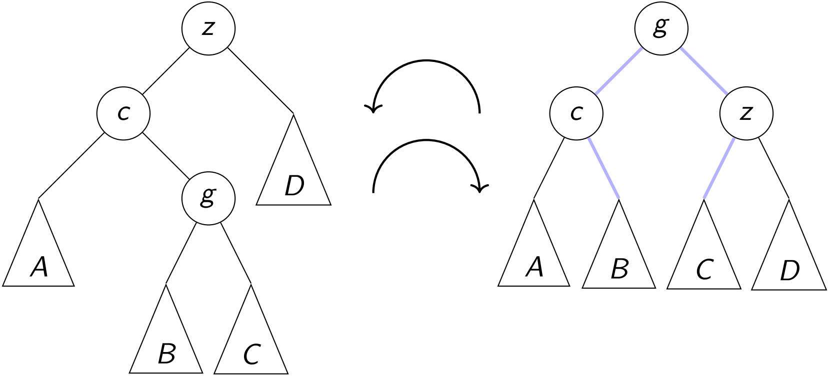
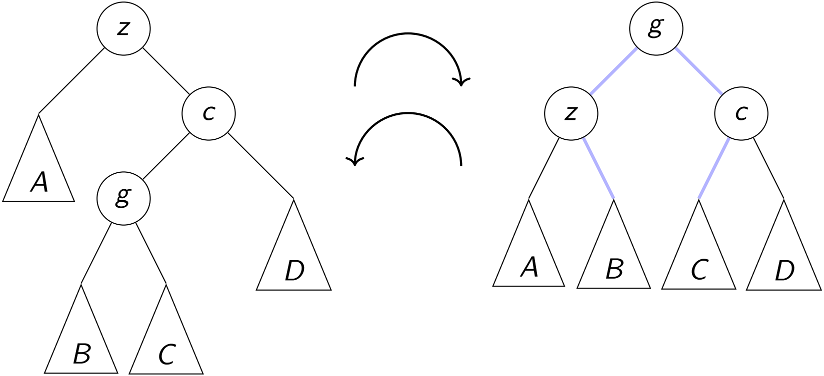


First, a left rotation at c.
Second, a right rotation at z.

Double Left Rotation

Symmetrically, there is a **double left rotation** on node z:



First, a right rotation at c.
Second, a left rotation at z.

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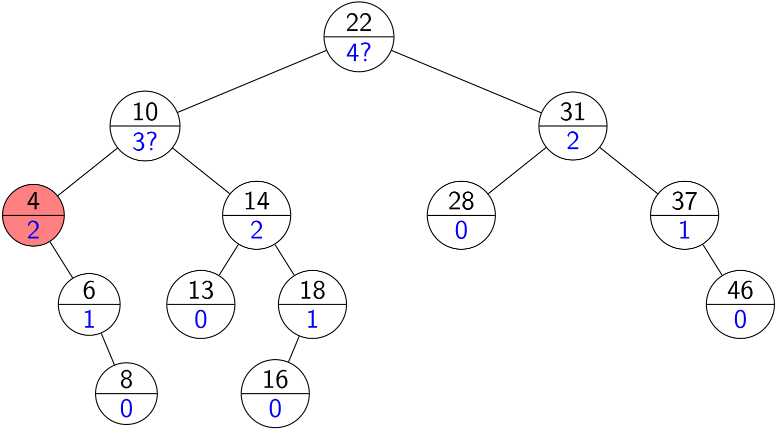
AVL Insertion Example revisited

Example: *AVL::insert*(8)



AVL insertion revisited

- Imbalance at z: do (single or double) rotation
- Choose c as child where subtree has bigger height.

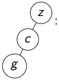
```
AVL::insert(k, v)
    z ← BST::insert(k, v)   // leaf where k is now stored
    while (z is not NULL)
        if (|z.left.height − z.right.height| > 1) then
            Let c be taller child of z
            Let g be taller child of c (so grandchild of z)
            restructure(g, c, z) // see later
            break        // can argue that we are done
        setHeightFromSubtrees(z)
        z ← z.parent
```

Can argue: For insertion *one* rotation restores all heights of subtrees.

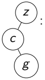
⇒ No further imbalances, can stop checking.

Fixing a slightly-unbalanced AVL tree

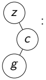
```
restructure($g, c, z$)
node $g$ is child of $c$ which is child of $z$
1.  $p \leftarrow z.parent$        // save for later
2.  case { left
```



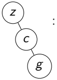
```
        :  // Right rotation
           $u \leftarrow$ rotate-right($z$)
```



```
        :  // Double-right rotation
           $z.$left $\overset{p}{\leftarrow}$ rotate-left($c$)
           $u \leftarrow$ rotate-right($z$)
```



```
        :  // Double-left rotation
           $z.$right $\overset{p}{\leftarrow}$ rotate-right($c$)
           $u \leftarrow$ rotate-left($z$)
```

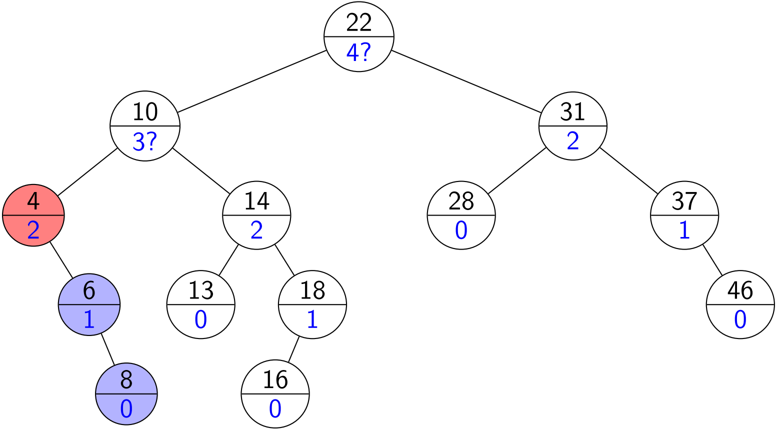


```
        :  // Left rotation
           $u \leftarrow$ rotate-left($z$)
3.  make $u$ the appropriate child of $p$ and return $u$
```

Rule: The middle key of g, c, z becomes the new root.

AVL Insertion Example revisited

Example: *AVL::insert*(8)



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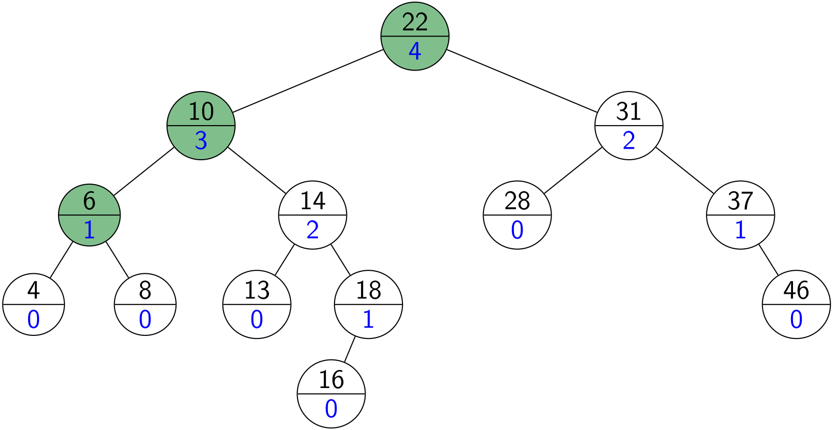
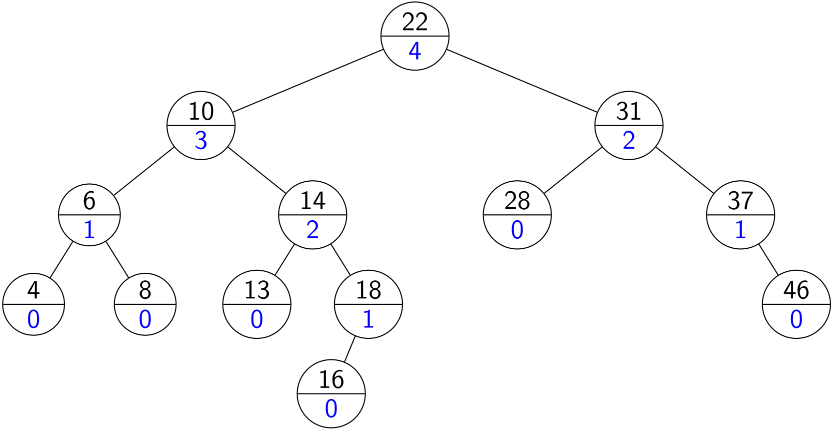


AVL Insertion: Second example

Example: *AVL::insert*(45)



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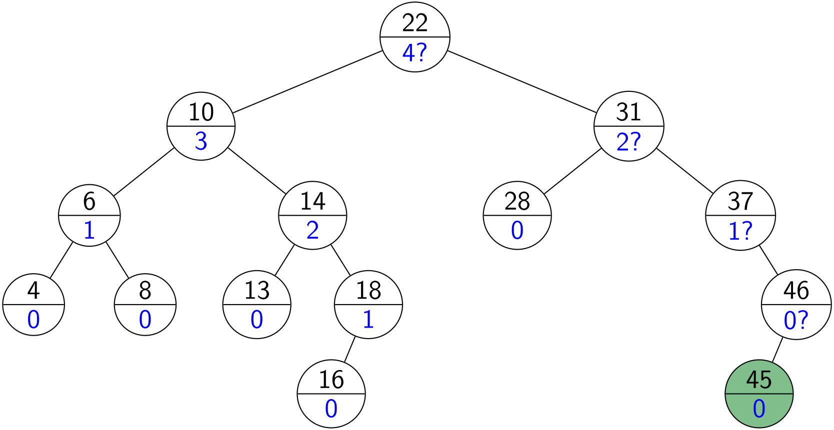

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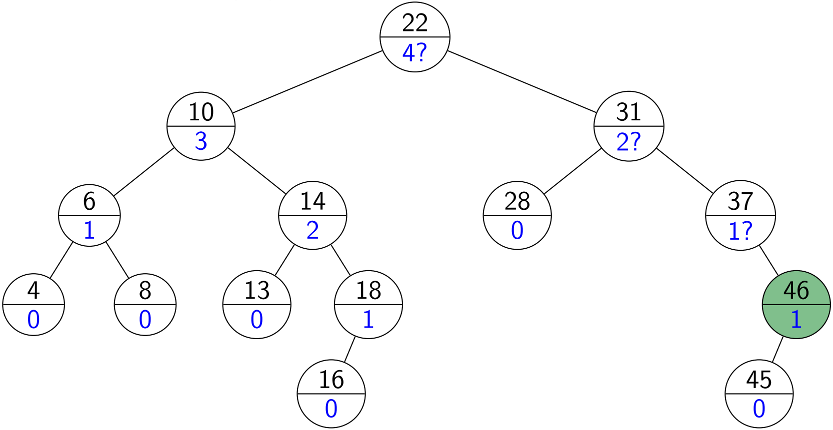

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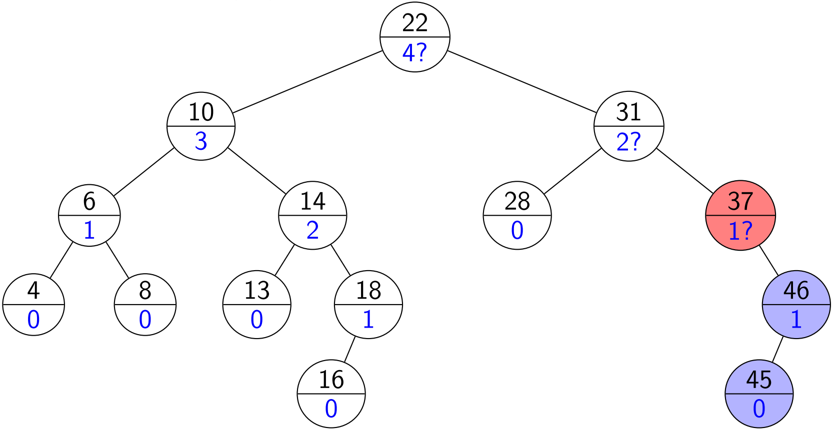

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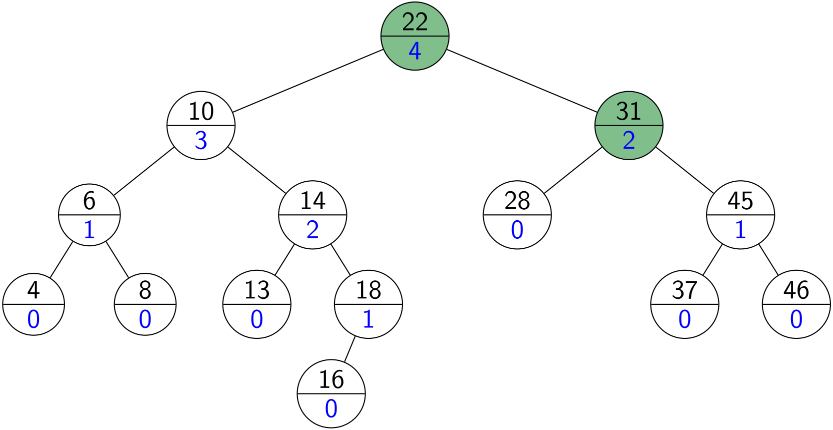


Outline

4. Dictionaries and Balanced Search Trees
   - ADT Dictionary
   - Binary Search Trees
   - AVL Trees
   - Insertion in AVL Trees
   - Restructuring a BST: Rotations
   - AVL insertion revisited
   - Deletion in AVL Trees

AVL Deletion

Remove the key k with *BST::delete*.

Find node where *structural* change happened.

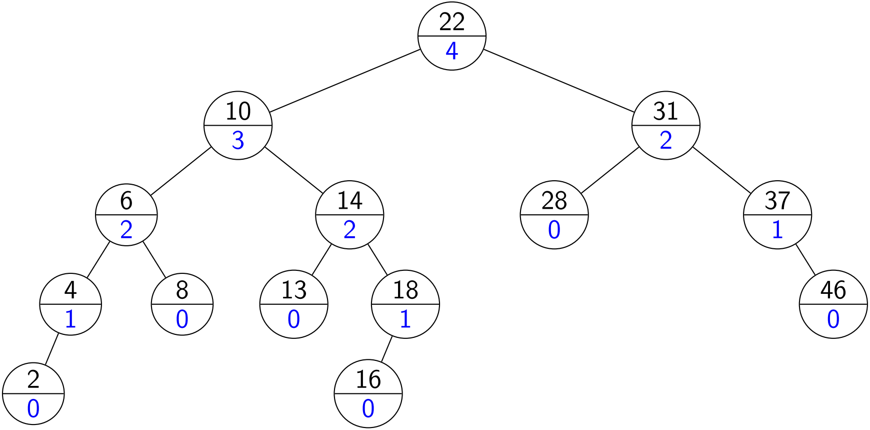
(This is not necessarily near the node that had k.)

Go back up to root, update heights, and rotate if needed.

```
AVL::delete(k)
z ← BST::delete(k)
// Assume z is the parent of the BST node that was removed
while (z is not NULL)
    if ($|z.left.height - z.right.height| > 1$) then
        Let c be taller child of z
        Let g be taller child of c (break ties to avoid double rotation)
        z ← restructure(g, c, z)
    // Always continue up the path
    setHeightFromSubtrees(z)
    z ← z.parent
```

AVL Deletion Example

Example: *AVL::delete*(22)



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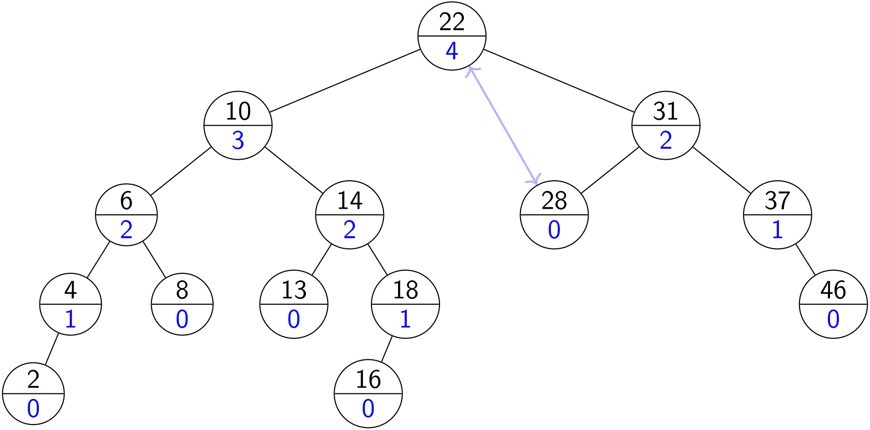
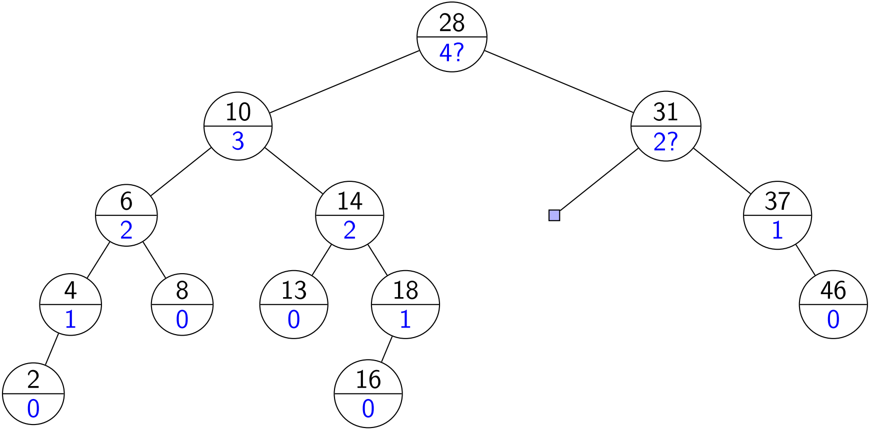
Example: *AVL::delete*(22)



AVL Deletion Example

Example: *AVL::delete*(22)



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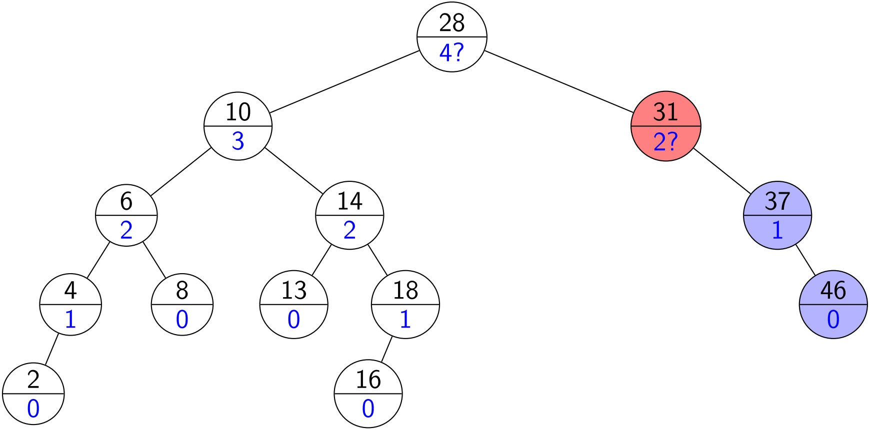
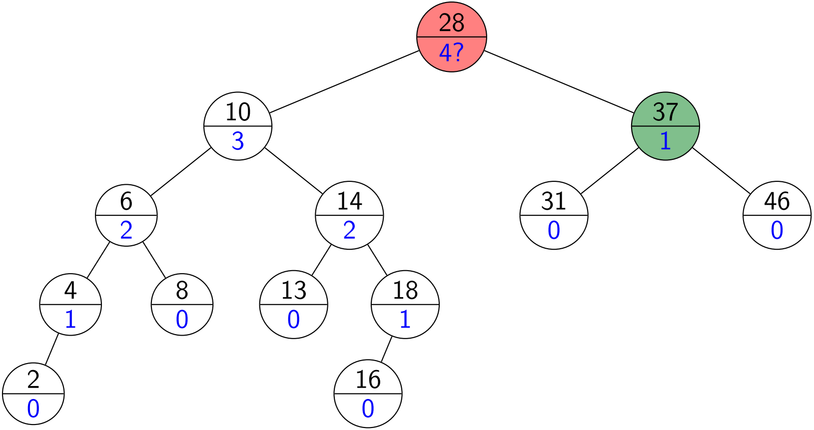
Example: *AVL::delete*(22)



AVL Deletion Example

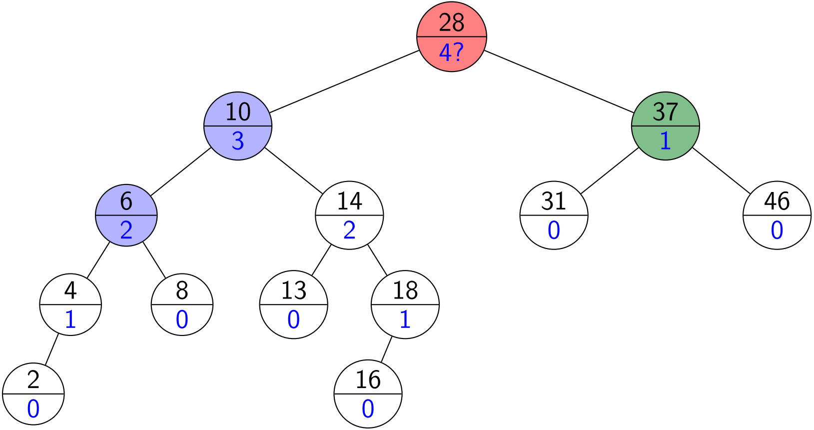
Example: *AVL::delete*(22)



A single *restructure* is not enough to restore all balances.

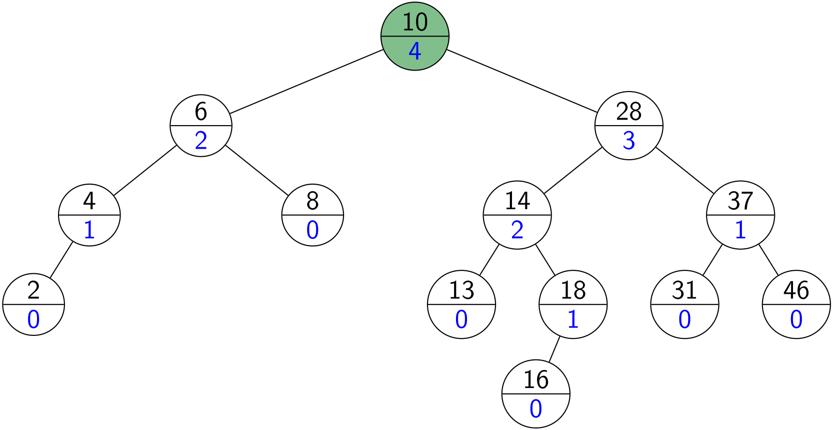
AVL Deletion Example

Example: *AVL::delete*(22) (cont'd)



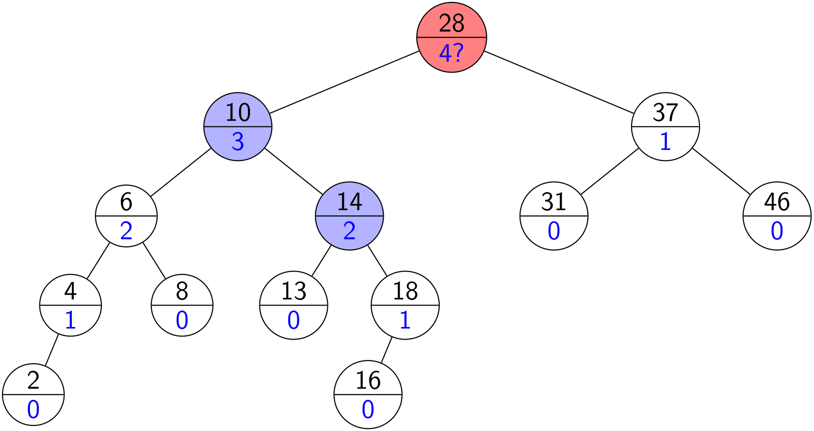
AVL Deletion Example

Example: *AVL::delete*(22) (cont'd)



AVL Deletion Example

Important: Ties *must* be broken to avoid double rotation.
Consider again the above example. If we applied double-rotation:



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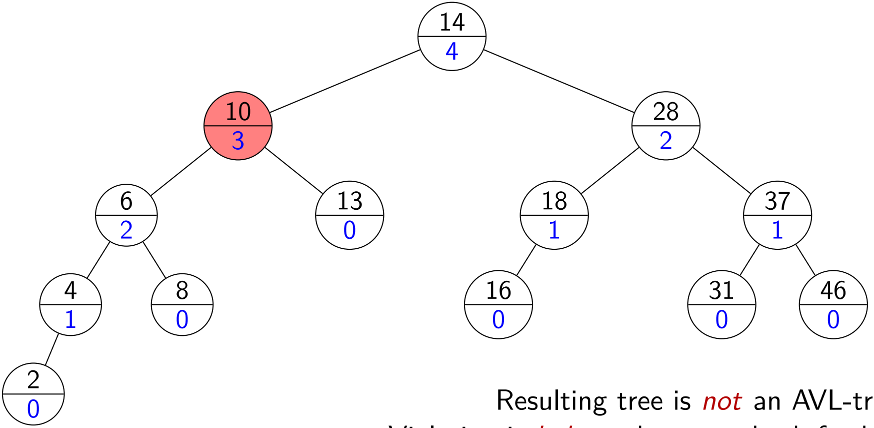

Resulting tree is *not* an AVL-tree.
Violation is *below* where we check further.

AVL Tree Summary

search: Just like in BSTs, costs $\Theta(height)$

insert: *BST::insert*, then check & update along path to new leaf

- total cost $\Theta(height)$
- *restructure* will be called *at most once*.

delete: *BST::delete*, then check & update along path to deleted node

- total cost $\Theta(height)$
- *restructure* may be called $\Theta(height)$ times.

Worst-case cost for all operations is $\Theta(height) = \Theta(\log n)$.

- In practice, the constant is quite large.
- Other realizations of ADT Dictionary are better in practice ($\rightarrow$ later)