CS 240 – Data Structures and Data Management

Module 3: Sorting, Average-case and Randomization

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Based on lecture notes by many previous cs240 instructors

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Outline

3. Sorting, Average-case and Randomization
   - Analyzing average-case run-time
   - SELECTION and *quick-select*
   - Randomized Algorithms
   - *quick-select* revisited
   - SORTING and *quick-sort*
   - Lower Bound for Comparison-Based Sorting
   - Non-Comparison-Based Sorting

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Average-case analysis

We will introduce (and solve) a new problem, and then analyze the *average-case run-time* of our algorithm.

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Recall definition of average-case run-time:

$$T_{\mathcal{A}}^{avg}(n) = \sum_{\text{instance } I \text{ of size } n} T_{\mathcal{A}}(I) \cdot (\text{relative frequency of } I)$$

For this module:

- Assume that the set $\mathcal{I}_n$ of size-n instances is finite (or can be mapped to a finite set in a natural way)
- Assume that all instances occur equally frequently

Then we can use the following *simplified formula*

$$T^{avg}(n) = \frac{\sum_{I:\text{size}(I)=n} T(I)}{\#\text{instances of size } n} = \frac{1}{|\mathcal{I}_n|} \sum_{I \in \mathcal{I}_n} T(I)$$

To learn how to analyze this, we will do simpler examples first.

A simple (contrived) example

silly-test(π, n)
π: a permutation of $\{0, \ldots, n{-}1\}$, stored as an array
1. **if** $\pi[0] = 0$ **then for** $j = 1$ to n **do** print 'bad case'
2. **else** print 'good case'

$$T^{avg}(n) \quad = \frac{1}{n!} \sum_{\pi \in \Pi_n} T(\pi) = \frac{1}{n!} \Big(\sum_{\substack{\pi \in \Pi_n \\ \text{in bad case}}} T(\pi) + \sum_{\substack{\pi \in \Pi_n \\ \text{in good case}}} T(\pi) \Big)$$

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- The remaining $n! - (n-1)!$ permutations have run-time c.

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$$T^{avg}(n) = \frac{1}{n!}\Big(\#\{\pi \in \Pi_n \text{ in bad case}\} \cdot cn + \#\{\pi \in \Pi_n \text{ in good case}\} \cdot c\Big)$$
$$= \frac{1}{n!}\Big((n-1)! \cdot cn + (n! - (n-1)!) \cdot c\Big) \le \frac{1}{n}cn + c = 2c \in O(1)$$

A second (not-so-contrived) example

```
all-0-test(w, n)
// test whether all entries of bitstring w[0..n−1] are 0
1.  if (n = 0) return true
2.  if (w[n−1] = 1) return false
3.  all-0-test(w, n−1)
```

(In real life, you would write this non-recursive.)

Define $T(w) = \#$ bit-comparisons (i.e., line 2) on input w. This is asymptotically the same as the run-time.

Worst-case run-time: Always go into the recursion until $n = 0$.
$T(n) = 1 + T(n-1) = 1 + 1 + \cdots + T(0) = n \in \Theta(n)$.

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Best-case run-time: Return immediately. $T(n) = 1 \in \Theta(1)$.

Average-case run-time?

Average-case run-time of *all-0-test*

Recall $T^{\text{avg}}(n) = \frac{1}{|\mathcal{B}_n|} \sum_{w \in \mathcal{B}_n} T(w)$. $\quad$ ($\mathcal{B}_n = \{$bitstrings of length $n\}$)

Recursive formula for one non-empty bitstring w:

$$T(w) = \begin{cases} 1 & \text{if } w[n-1] = 1 \\ 1 + T(\underbrace{w[0..n-2]}_{\text{length } n-1}) & \text{otherwise} \end{cases}$$

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Natural guess for the recursive formula for $T^{avg}(n)$:

$$T^{avg}(n) = \underbrace{\tfrac{1}{2}}_{\substack{\text{half have} \\ w[n-1]=1}} \cdot 1 + \underbrace{\tfrac{1}{2}}_{\substack{\text{half have} \\ w[n-1]=0}} (1 + T^{???}(n-1))$$

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- This holds with $\leq$ (but is useless) if '???' is 'worst'.
- This is *not obvious* if '???' is 'avg'.

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&= 1 + \frac{1}{|\mathcal{B}_n|} \sum_{w' \in \mathcal{B}_{n-1}} T(w') \\
&= 1 + \frac{|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|} \frac{1}{|\mathcal{B}_{n-1}|} \sum_{w' \in \mathcal{B}_{n-1}} T(w') = 1 + \frac{1}{2} T^{avg}(n-1)
\end{aligned}
$$

This recursion resolves to $T^{avg}(n) \in O(1)$.

Average-case analysis and recursions

Why can't we always write 'avg' for '???' in $T^{avg}(n) = 1 + \frac{1}{2} T^{???}(n-1)$?

Consider the following (contrived) example:

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- But observe that now $T(w) = \begin{cases} 1 & \text{if } w[n-1] = 1 \\ n & \text{if } w[n-1] = 0 \end{cases}$.
- So $T^{avg}(n) = \frac{1}{2} + \frac{n}{2} \in \Theta(n)$. The 'obvious' recursion did not hold.

Average-case analysis is highly non-trivial for recursive algorithms.

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The Selection Problem

We saw Selection: Given an array A of n numbers, and $0 \leq k < n$, find the element that would be at position k of the sorted array.

0	1	2	3	4	5	6	7	8	9
30	60	10	0	50	80	90	10	40	70

select(3) should return 30.

Selection can be done with heaps in time $\Theta(n + k \log n)$ (module 2), or even $\Theta(n + k \log k)$ (non-trivial exercise).

Special case: MedianFinding = Selection with $k = \lfloor \frac{n}{2} \rfloor$. With previous approaches, this takes time $\Theta(n \log n)$, no better than sorting.

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Question: Can we do selection in linear time?
Yes! We will develop algorithm *quick-select* below.

The encountered sub-routines will also be useful otherwise.

Crucial Subroutines

quick-select and the related *quick-sort* rely on two subroutines:

- *choose-pivot*(A): Return an index p in A. We will use the **pivot-value** $v \leftarrow A[p]$ to rearrange the array.

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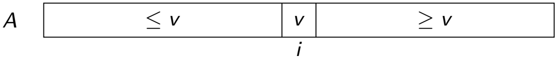
- *partition*(A, p): Rearrange A and return **pivot-index** i so that
  - the pivot-value v is in $A[i]$,
  - all items in $A[0, \ldots, i-1]$ are $\leq v$, and
  - all items in $A[i+1, \ldots, n-1]$ are $\geq v$.



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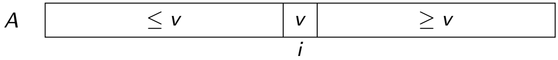

- p = index of pivot-value before *partition* (we choose it)
  i = index of pivot-value after *partition* (no control)

Partition Algorithm

Conceptually easy linear-time implementation:

partition(A, p)
A: array of size n, p: integer s.t. $0 \leq p < n$

Create empty lists *smaller*, *equal* and *larger*.
$v \leftarrow A[p]$
for each element x in A **do**
  if $x < v$ **then** *smaller.append*(x)
  else if $x > v$ **then** *larger.append*(x)
  else *equal.append*(x).
$i \leftarrow smaller.size$
$j \leftarrow equal.size$
Overwrite $A[0 \ldots i{-}1]$ by elements in *smaller*
Overwrite $A[i \ldots i{+}j{-}1]$ by elements in *equal*
Overwrite $A[i{+}j \ldots n{-}1]$ by elements in *larger*
return i

More challenging: partition **in place** (with $O(1)$ auxiliary space).

Efficient In-Place partition (Hoare) - Example

Idea: Keep swapping the outer-most wrongly-positioned pairs.

i=-1	0	1	2	3	4	5	6	7	8	j=9
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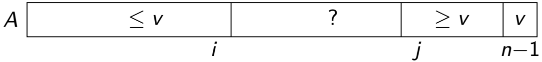
Efficient In-Place partition (Hoare)

Loop invariant:



partition(A, p)
A: array of size n, p: integer s.t. $0 \le p < n$

```
1.  swap(A[n−1], A[p])
2.  i ← −1,   j ← n−1,   v ← A[n−1]
3.  loop
4.        do i ← i+1 while A[i] < v
5.        do j ← j−1 while j > i and A[j] > v
6.        if i ≥ j then break    (goto 9)
7.        else swap(A[i], A[j])
8.  end loop
9.  swap(A[n−1], A[i])
10. return i
```

Running time: $\Theta(n)$.

Observe: n **key-comparisons** (comparing two input-items).

quick-select Algorithm

SELECTION: Want item m such that (after rearranging A) we have

$\leq m$	m	$\geq m$
	k	

```
quick-select(A, k)
A: array of size n,   k: integer s.t. 0 ≤ k < n
    p ← choose-pivot(A)
    i ← partition(A, p)
    if i = k then return A[i]
    else if i > k then return quick-select(A[0 ... i−1], k)
    else if i < k then return quick-select(A[i+1 ... n−1], k − (i+1))
```

Idea: After partition have

$\leq v$	v	$\geq v$
	i	

Where is m if $k = i$? If $k < i$? If $k > i$?

Analysis of *quick-select*

Let $T(A, k)$ be the number of key-comparisons in a size-n array A with parameter k. (This is asymptotically the same as the run-time.)

partition uses n key-comparisons.

Worst-case run-time:

- Sub-array always gets smaller, so $\leq n$ recursions.
  Each takes $\leq n$ comparisons $\Rightarrow O(n^2)$ time.
- This is tight: If pivot-value is always the maximum and $k = 0$
  $T^{\text{worst}}(n, 0) \geq n + (n-1) + (n-2) + \cdots + 1 \in \Omega(n^2)$

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  Each takes $\leq n$ comparisons $\Rightarrow O(n^2)$ time.
- This is tight: If pivot-value is always the maximum and $k = 0$
  $T^{\text{worst}}(n, 0) \geq n + (n-1) + (n-2) + \cdots + 1 \in \Omega(n^2)$

Best-case run-time: First chosen pivot could be the kth element
No recursive calls; $T^{\text{best}}(n, k) = n \in \Theta(n)$

Analysis of *quick-select*

Let $T(A, k)$ be the number of key-comparisons in a size-n array A with parameter k. (This is asymptotically the same as the run-time.)

partition uses n key-comparisons.

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Average case analysis? Doing this directly would be *very* complicated.
Instead we will do it via a detour into a randomized version.

Outline

3. Sorting, Average-case and Randomization
   - Analyzing average-case run-time
   - SELECTION and *quick-select*
   - Randomized Algorithms
   - *quick-select* revisited
   - SORTING and *quick-sort*
   - Lower Bound for Comparison-Based Sorting
   - Non-Comparison-Based Sorting

Randomized algorithms

- A **randomized algorithm** is one which relies on some random numbers in addition to the input.
  - Doing randomization is often a good idea if an algorithm has bad worst-case time but seems to perform much better on most instances.
  - It can also (with restrictions) be used to bound the avg-case run-time.
- The run-time will depend on the input and the random numbers used.

(Computers cannot generate randomness. We assume that there exists a *pseudo-random number generator (PRNG)*, a deterministic program that uses an initial value or *seed* to generate a sequence of seemingly random numbers. The quality of randomized algorithms depends on the quality of the PRNG!)

- **Goal:** Shift the dependency of run-time from what we can't control (the input) to what we *can* control (the random numbers).

  No more bad instances, just unlucky numbers.

Example (again very contrived)

```
randomized-all-0-test(w, n)
w: array of size at least n that stores bits
 1.  if n = 0 return true
 2.  if (random(2)=0) then
             w[n−1] = 1 − w[n−1]      // this is the only change
 3.  if w[n−1] = 1 return false
 4.  randomized-all-0-test(w, n−1)
```

This is *all-0-test*, except that we flip last bit based on a coin toss.

We assume the existence of a function *random*(n) that returns an integer uniformly from $\{0, 1, 2, \ldots, n-1\}$. So $Pr(random(2) = 0) = \frac{1}{2}$.

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In each recursion, we use the outcome $x \in \{0, 1\}$ of one coin toss.
We return without recursing if $x = w[n-1]$ (this has probability $\frac{1}{2}$).

Expected run-time

The run-time of the algorithm now depends on the random numbers.

Define $T_{\mathcal{A}}(I, R)$ to be the run-time of a randomized algorithm $\mathcal{A}$ for an instance I and the sequence R of outcomes of random trials.

The **expected run-time $T^{exp}(I)$ for instance I** is the expected value:

$$T^{exp}(I) = \mathbf{E}[T(I, R)] = \sum_{R} T(I, R) \cdot \Pr(R)$$

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We can still have good luck or bad luck, so occasionally we also discuss the very worst that could happen, i.e., $\max_I \max_R T(I, R)$.

Expected run-time of *randomized-all-0-test*

Define $T(w, R) := \#$ bit-comparisons used on input w if the random outcomes are R. (This is proportional to the run-time.)

- The random outcomes R consist of two parts $R = \langle x, R' \rangle$:
  - x: outcome of first coin toss
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- Recursive formula for one instance:

$$T(w, R) = T(w, \langle x, R' \rangle) = \begin{cases} 1 & \text{if } x = w[n{-}1] \\ 1 + T(w[0..n{-}2], R') & \text{otherwise} \end{cases}$$

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- Natural guess for the recursive formula for $T^{exp}(n)$:

$$T^{exp}(n) = \underbrace{\tfrac{1}{2}}_{\Pr(x=w[n-1])} \cdot 1 + \underbrace{\tfrac{1}{2}}_{\Pr(x \neq w[n-1])} (1 + T^{exp}(n-1)) = 1 + \tfrac{1}{2} T^{exp}(n-1)$$

Expected run-time of *randomized-all-0-test*

In contrast to average-case analysis, the natural guess usually is correct for the expected run-time.

Proof for *randomized-all-0-test*:

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&\le 1 + \tfrac{1}{2} \max_{w' \in \mathcal{B}_{n-1}} T^{exp}(w') = 1 + \tfrac{1}{2} T^{\exp}(n{-}1) \quad \text{holds for } \textit{all } w
\end{aligned}
$$

Therefore $T^{exp}(n) = \max_{w \in \mathcal{B}_n} T^{exp}(w) \le 1 + \frac{1}{2} T^{exp}(n{-}1)$

Expected run-time of *randomized-all-0-test*

- We had $T^{exp}_{\textit{rand-all-0-test}}(n) \le 1 + \frac{1}{2} T^{exp}_{\textit{rand-all-0-test}}(n-1)$
- We earlier had $T^{avg}_{\textit{all-0-test}}(n) \le 1 + \frac{1}{2} T^{avg}_{\textit{all-0-test}}(n-1)$
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- Recall: *randomized-all-0-test* was very similar to *all-0-test*
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- Is it a coincidence that the two recursive formulas are the same?
  Or does the expected time of a randomized version always have something to do with the average-case time?

- Not in general! (It depends how we randomize.)
- Yes if the randomization is a *shuffle* (choose instance randomly).

Avg-case run-time via expected run-time

Consider the following randomization of a deterministic algorithm $\mathcal{A}$.

> *shuffled-$\mathcal{A}$*(n)
> 1. Among all instances $\mathcal{I}_n$ of size n for $\mathcal{A}$, choose I randomly
> 2. $\mathcal{A}(I)$

(*shuffled-$\mathcal{A}$* usually does not solve what $\mathcal{A}$ solves)

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- So the average-case run-time of $\mathcal{A}$ is the same as this **run-time of $\mathcal{A}$ on randomly chosen input**.
- This gives us a different way to compute $T^{avg}_{\mathcal{A}}(n)$.

Avg-case run-time via expected run-time

Example: *all-0-test* (rephrased with for-loops):

shuffled-all-0-test(n)

1. **for** $(i = n{-}1;\, i \geq 0;\, i{-}{-})$ **do**
2. $\quad w[i] \leftarrow$ *random*(2)
3. **for** $(i = n{-}1;\, i \geq 0;\, i{-}{-})$ **do**
4. $\quad$ **if** $(w[i] = 1)$ **return** `false`
5. **return** `true`

randomized-all-0-test(w, n)

1. **for** $(i = n{-}1;\, i \geq 0;\, i{-}{-})$ **do**
2. $\quad$ **if** $($*random*$(2){=}0)$ **then**
   $\qquad w[i] = 1 - w[i]$
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- These algorithms are not quite the same.
    - Randomization outside resp. inside the for-loop.
- But this does not matter for the expected number of bit-comparisons.
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- So $T^{avg}_{\textit{all-0-test}}(n) = T^{exp}_{\textit{shuffled-all-0-test}}(n) = T^{exp}_{\textit{rand-all-0-test}}(n) \in O(1)$

  can be deduced without analyzing $T^{avg}_{\textit{all-0-test}}(n)$ directly.

Summary: Average-case run-time vs. expected run-time

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average-case run-time	expected run-time
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average over instances	weighted average over random outcomes
(usually) applied to a deterministic algorithm	applied only to a randomized algorithm

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There is a relationship *only* if the randomization effectively achieves 'choose the input instance randomly'.

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Average-case analysis of quick-select

Recall *quick-select* (with *choose-pivot*$(A) = n - 1$):

quick-select(A, k)
1. $i \leftarrow$ *partition*$(A, n{-}1)$
2. **if** $i = k$ **then return** $A[i]$
3. **else if** $i > k$ **then** *quick-select*$(A[0 \ldots i{-}1], k)$
4. **else if** $i < k$ **then** *quick-select*$(A[i{+}1 \ldots n{-}1], k - (i{+}1))$

For analyzing the average-case run-time, we make two **assumptions**:

- All input-items are distinct.
  - This can be forced using tie-breakers.
- All possible orders of the input-items occur equally often.
  - This is not completely realistic (mostly-sorted orders are more common).
  - But we cannot do average-case analysis without it.

Randomizing quick-select: Shuffling

Goal: Create a randomized version of *quick-select*.

- This will give a proof of the avg-case run-time of *quick-select*.
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First idea: Shuffle the input, then do *quick-select*.

shuffled-quick-select(A, k)
1. **for** $(j \leftarrow 1$ to $n-1)$ **do** *swap*$(A[j], A[$*random*$(j+1)])$ `// shuffle`
2. *quick-select*(A, k)

- Shuffling (permuting) the input-array is (by assumption) equivalent to randomly choosing an input instance.
- So we know $T^{avg}_{\textit{quick-select}}(n) = T^{exp}_{\textit{shuffled-quick-select}}(n)$

(Recall: $T(\cdot)$ counts key-comparisons, so shuffling is free.)

Randomizing quick-select: Random Pivot

Second idea: Do the shuffling inside the recursion.
(Equivalently: Randomly choose which value is used for the pivot.)

randomized-quick-select(A, k)
1. swap $A[n{-}1]$ with $A[random(n)]$
2. $i \leftarrow$ *partition*$(A, n{-}1)$
3. **if** $i = k$ **then return** $A[i]$
4. **else if** $i > k$ **then**
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- $i \leftarrow$ *partition*$(A, n-1)$
- **if** $i = k$ **then return** $A[i]$
- **else if** $i > k$ **then**
  - **return** *randomized-quick-select*$(A[0 \ldots i-1], k)$
- **else if** $i < k$ **then**
  - **return** *randomized-quick-select*$(A[i+1 \ldots n-1], k - (i+1))$

- $T^{exp}_{\text{rand.-quick-select}}(n) = T^{exp}_{\text{shuffled-quick-select}}(n)$.

(This is not completely obvious, but believable. No proof.)

Expected run-time of *randomized-quick-select*

Let $T(A, k, R) = \#$ key-comparisons of *randomized-quick-select* on input $\langle A, k \rangle$ if the random outcomes are R. (This is proportional to the run-time.)

- Write random outcomes R as $R = \langle i, R' \rangle$ (where 'i' stands for 'the first random number was such that the pivot-index is i')
- Observe: $\Pr(\text{pivot-index is } i) = \frac{1}{n}$
- We recurse in an array of size i or $n-i-1$ (or not at all)

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- Recursive formula for one instance (and fixed $R = \langle i, R'\rangle$):

$$T(A, k, \langle i, R'\rangle) = n + \begin{cases} T(\text{ size-}i \text{ array}, k, R') & \text{if } i > k \\ T(\text{ size-}(n-i-1) \text{ array}, k-i-1, R') & \text{if } i < k \\ 0 & \text{otherwise} \end{cases}$$

Analysis of *randomized-quick-select*

As for *rand-all-0-test*: over all R, the recursions can use T^{exp}(array-size).

$$T^{exp}(A,k) \left(\begin{aligned} &= \sum_R P(R)\cdot T(\langle A,k\rangle,R) = \sum_{i=0}^{n-1}\sum_{R'} P(i)\cdot P(R')\cdot T(\langle A,k\rangle,\langle i,R'\rangle) \\ &= \frac{1}{n}\sum_{i=0}^{k-1}\sum_{R'} P(R')\Big(n + T(\langle A[i{+}1..n{-}1],k{-}i{-}1\rangle,R')\Big) \\ &\qquad + \underbrace{\frac{1}{n}\cdot n}_{i=k} \quad + \frac{1}{n}\sum_{i=k+1}^{n-1}\sum_{R'} P(R')\Big(n + T(\langle A[0..i{-}1,k\rangle,R')\Big) \\ &= n + \frac{1}{n}\sum_{i=0}^{k-1}\sum_{R'} P(R')T(\langle A[i{+}1..n{-}1],k{-}i{-}1\rangle,R') \\ &\qquad\qquad + \frac{1}{n}\sum_{i=k+1}^{n-1}\sum_{R'} P(R')T(\langle A[0..i{-}1,k\rangle,R') \\ &= n + \frac{1}{n}\sum_{i=0}^{k-1}\underbrace{T^{\mathrm{exp}}(\langle A[i{+}1..n{-}1],k{-}i{-}1\rangle)}_{\le T^{\mathrm{exp}}(n-i-1)} + \frac{1}{n}\sum_{i=k+1}^{n-1}\underbrace{T^{\mathrm{exp}}(\langle A[0..i{-}1],k\rangle)}_{\le T^{\mathrm{exp}}(i)} \end{aligned} \right) \text{tedious but straightforward}$$

$$\le n + \frac{1}{n}\sum_{i=0}^{n-1}\max\{T^{exp}(i), T^{exp}(n{-}i{-}1)\}$$ *independent* of A, k

Analysis of *randomized-quick-select*

In summary, the expected run-time of *randomized-quick-select* satisfies:

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Claim: This recursion resolves to $O(n)$.

Summary of SELECTION

- *randomized-quick-select* has expected run-time $\Theta(n)$.
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- So the expected run-time of *shuffled-quick-select* is $\Theta(n)$.
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- *randomized-quick-select* is generally the fastest solution to SELECTION.

- There exists a variation that solves SELECTION with worst-case run-time $\Theta(n)$, but it uses double recursion and is slower in practice. ($\rightarrow$ *cs*341, maybe)

Outline

3 Sorting, Average-case and Randomization
- Analyzing average-case run-time
- SELECTION and *quick-select*
- Randomized Algorithms
- *quick-select* revisited
- SORTING and *quick-sort*
- Lower Bound for Comparison-Based Sorting
- Non-Comparison-Based Sorting

quick-sort

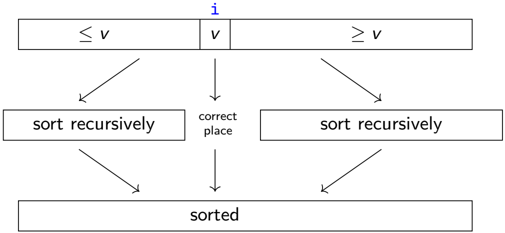
Hoare developed *partition* and *quick-select* in 1960.

He also used them to *sort* based on partitioning:

quick-sort(A)

A: array of size n

1. **if** $n \leq 1$ **then return**
2. $p \leftarrow$ *choose-pivot*(A)
3. $i \leftarrow$ *partition*(A, p)
4. *quick-sort*$(A[0, 1, \ldots, i-1])$
5. *quick-sort*$(A[i+1, \ldots, n-1])$



quick-sort analysis

Set $T(A) := \#$ of key-comparison for *quick-sort* in array A.

Worst-case run-time: $\Theta(n^2)$

- Sub-arrays get smaller $\Rightarrow \leq n$ levels of recursions
- On each level there are $\leq n$ items in total $\Rightarrow \leq n$ key-comparisons
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Best-case run-time: $\Theta(n \log n)$

- If pivot-index is always in the middle, then we recurse in two sub-arrays of size $\leq n/2$.
- $T(n) \leq n + 2T(n/2) \in O(n \log n)$ exactly as for *merge-sort*
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Average-case run-time? We again prove this via randomization.

Randomizing quick-sort

randomized-quick-sort(A)

```
if n ≤ 1 then return
p ← random(n)
i ← partition(A, p)
randomized-quick-sort(A[0, 1, ..., i−1])
randomized-quick-sort(A[i+1, ..., n−1])
```

Observe: $\Pr(\text{pivot has index } i) = \frac{1}{n}$

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Assume the random output was such that the pivot-index is i:

- We use n comparisons in *partition*.
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- We recurse in two arrays, of size i and $n{-}i{-}1$

This implies

$$T^{exp}(n) = \underbrace{\ldots \; = \; \ldots \; \leq \; \ldots}_{\text{long but straightforward}} = n + \frac{1}{n} \sum_{i=0}^{n-1} \Big(T^{exp}(i) + T^{exp}(n{-}i{-}1) \Big)$$

Expected run-time of *randomized-quick-sort*

$$T^{exp}(n) \leq n + \frac{1}{n}\sum_{i=0}^{n-1}\Big(T^{exp}(i) + T^{exp}(n-i-1)\Big) = n + \frac{2}{n}\sum_{i=1}^{n-1} T^{exp}(i)$$

(since $T(0) = 0$)

Claim: $T^{exp}(n) \in O(n \log n)$.

Proof:

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Summary of *quick-sort*

- *randomized-quick-sort* has expected run-time $\Theta(n \log n)$.
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- Can show: This has the same expected run-time as *quick-sort* on randomly chosen input (no details)
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- Auxiliary space?
  - Each nested recursion-call requires $\Theta(1)$ space on the call stack.
  - As described, *quick-sort*/*randomized-quick-sort* use $\Omega(n)$ nested recursion-calls in the worst case.
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- There are numerous tricks to improve *randomized-quick-sort*
- With these, this is in practice the fastest solution to SORTING (but *not* in theory).

quick-sort with tricks

randomized-quick-sort-improved(A, n)

```
Initialize a stack S of index-pairs with { (0, n−1) }
while S is not empty
    (ℓ, r) ← S.pop()
    while (r−ℓ+1 > 10) do
        p ← ℓ + random(ℓ−r+1)
        i ← partition(A, ℓ, r, p)
        if (i−ℓ > r−i) do
            S.push( (ℓ, i−1) )
            ℓ ← i+1
        else
            S.push( (i+1, r) )
            r ← i−1
insertion-sort(A)
```

Outline

3. Sorting, Average-case and Randomization
   - Analyzing average-case run-time
   - Selection and *quick-select*
   - Randomized Algorithms
   - *quick-select* revisited
   - Sorting and *quick-sort*
   - Lower Bound for Comparison-Based Sorting
   - Non-Comparison-Based Sorting

Lower bounds for sorting

We have seen many sorting algorithms:

Sort	Running time	Analysis
selection-sort	$\Theta(n^2)$	worst-case
insertion-sort	$\Theta(n^2)$ $\Theta(n)$	worst-case best-case
merge-sort	$\Theta(n \log n)$	worst-case
heap-sort	$\Theta(n \log n)$	worst-case
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Question: Can one do better than $\Theta(n \log n)$ running time?
Answer: Yes and no! *It depends on what we allow*.

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	$\Theta(n)$	best-case
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heap-sort	$\Theta(n \log n)$	worst-case
quick-sort	$\Theta(n \log n)$	average-case
randomized-quick-sort	$\Theta(n \log n)$	expected

Question: Can one do better than $\Theta(n \log n)$ running time?
Answer: Yes and no! *It depends on what we allow*.

- No: Comparison-based sorting lower bound is $\Omega(n \log n)$.
- Yes: Non-comparison-based sorting can achieve $O(n)$ (under restrictions!). ($\rightarrow$ later)

Lower bound for sorting in the comparison model

All algorithms so far are **comparison-based**: Data is accessed only by

- comparing two elements (a *key-comparison*)
- moving elements around (e.g. copying, swapping)

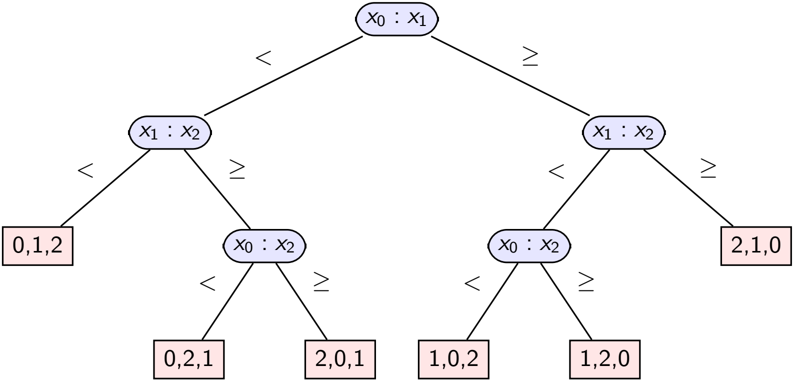
Theorem. Any *comparison-based* sorting algorithm requires in the worst case $\Omega(n \log n)$ comparisons to sort n distinct items.

Proof.

Decision trees

Any comparison-based algorithms can be expressed as **decision tree**.

To sort $\{x_0, x_1, x_2\}$:



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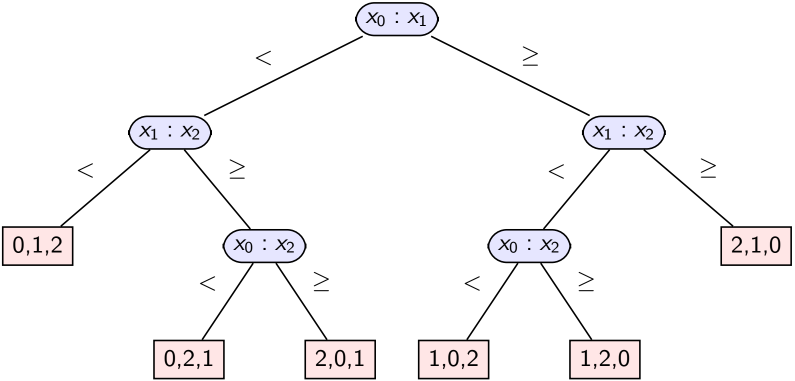
To sort $\{x_0, x_1, x_2\}$:

Example: $\{x_0{=}4, x_1{=}2, x_2{=}7\}$



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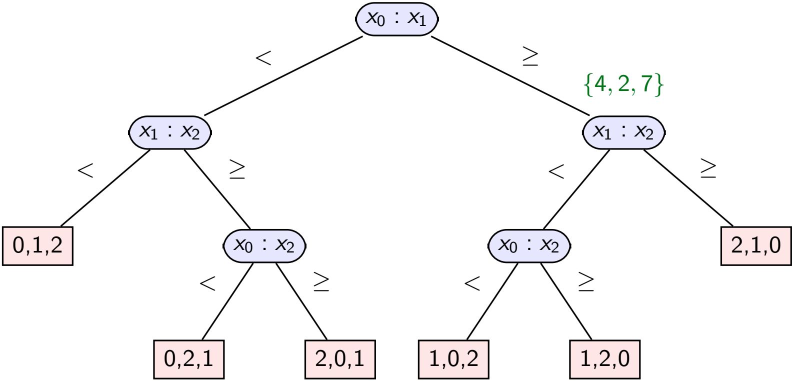

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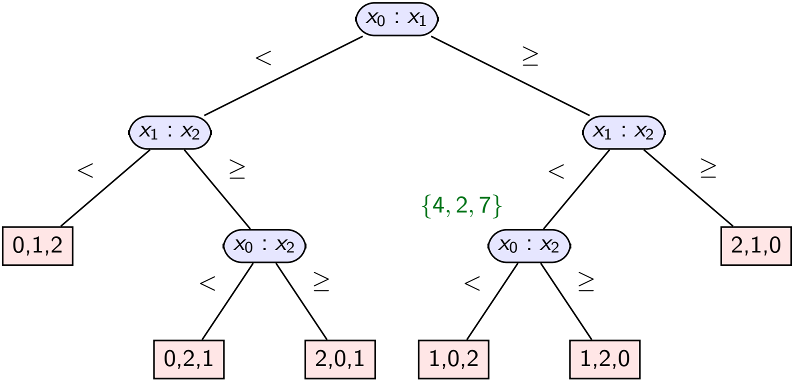

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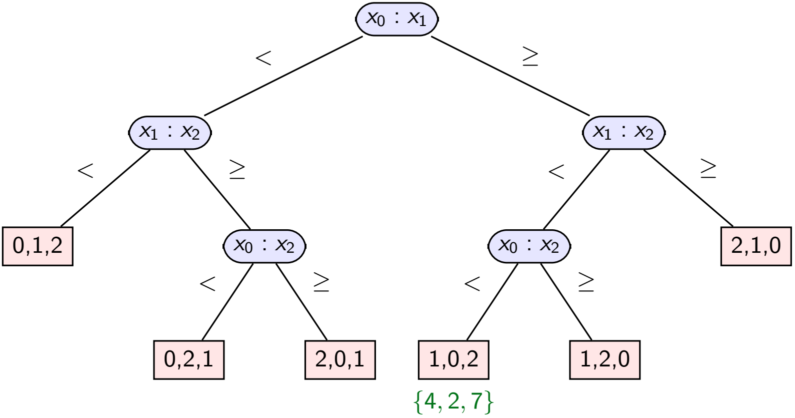


Output: $\{4, 2, 7\}$ has sorting permutation $\langle 1, 0, 2 \rangle$

(i.e., $x_1{=}2 \leq x_0{=}4 \leq x_2{=}7$)

Outline

Non-Comparison-Based Sorting

- Assume keys are numbers in base R (R: **radix**)
  - So all digits are in $\{0, \ldots, R-1\}$
  - $R = 2, 10, 128, 256$ are the most common, but R need not be constant

Example ($R = 4$):

123	230	21	320	210	232	101

- Assume all keys have the same number m of digits.
  - Can achieve after padding with leading 0s.
  - In typical computers, $m = 32$ or $m = 64$, but m need not be constant

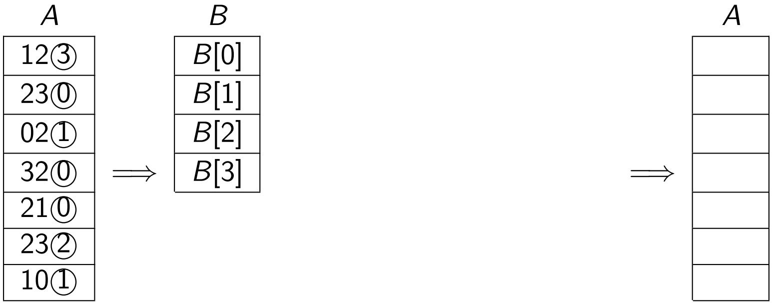
Example ($R = 4$):

123	230	021	320	210	232	101

- Can sort based on individual digits.
  - How to sort 1-digit numbers?
  - How to sort multi-digit numbers based on this?

(Single-digit) *bucket-sort*

Sort array A by last digit:



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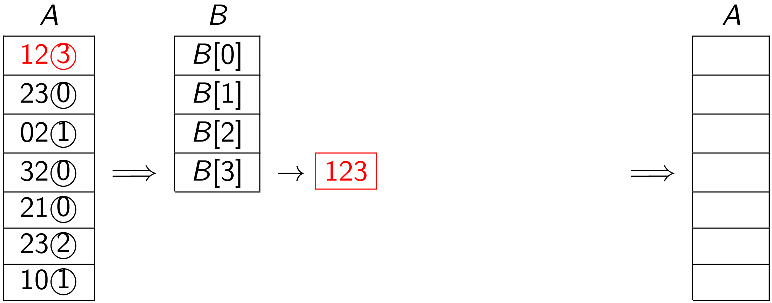

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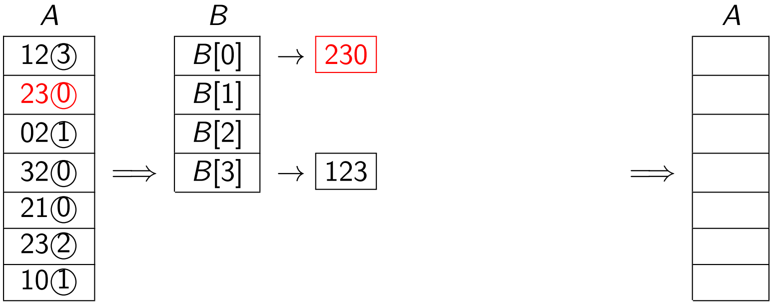

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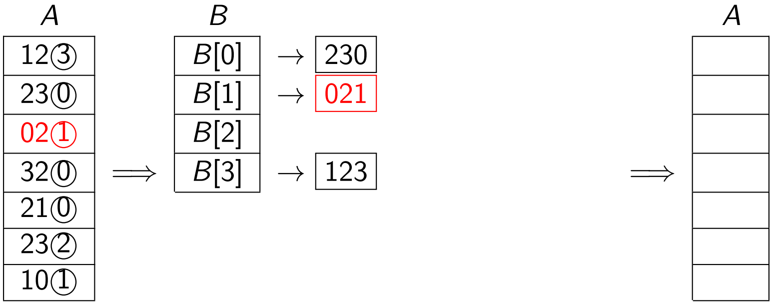

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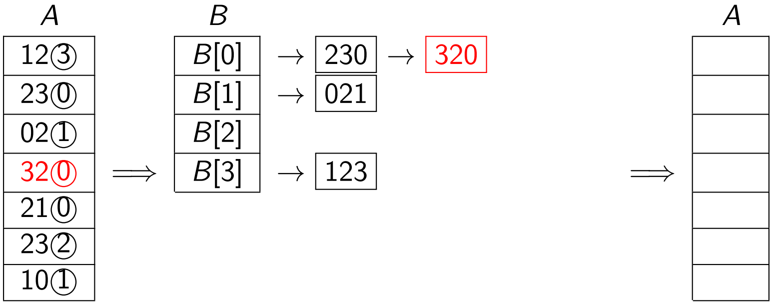

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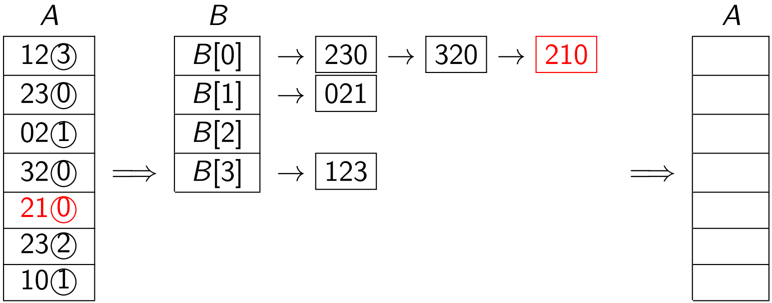

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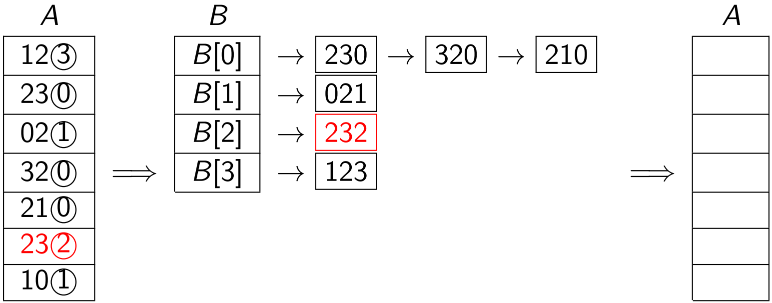

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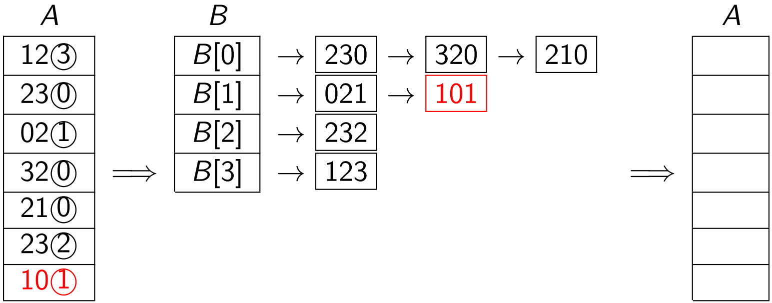

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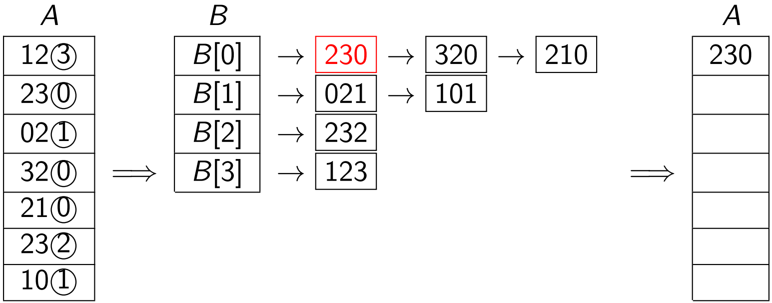

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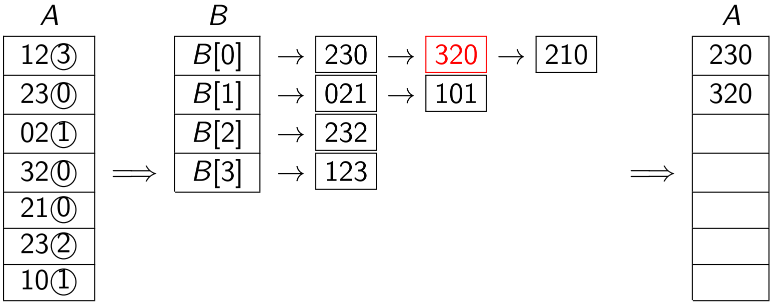

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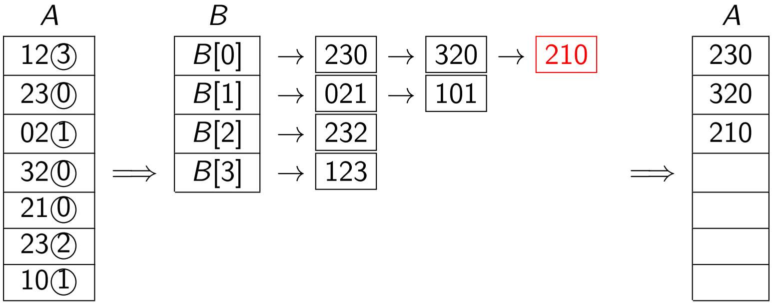

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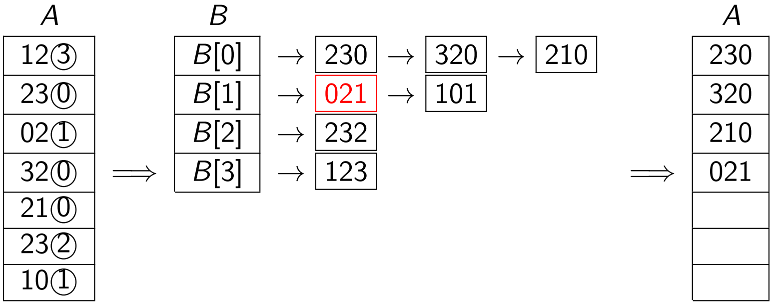

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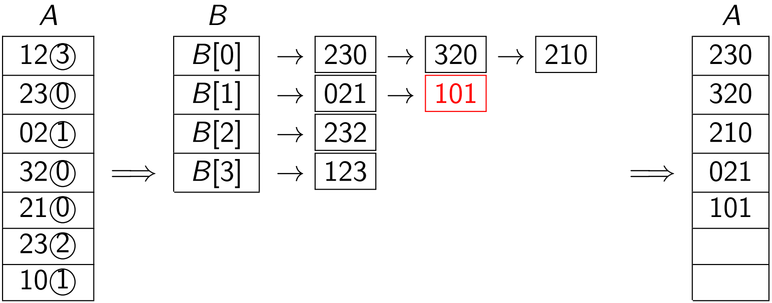

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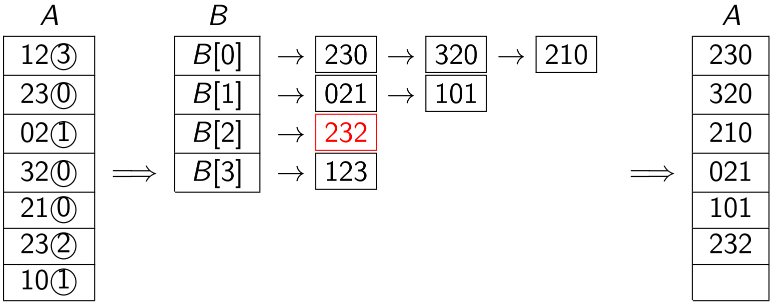

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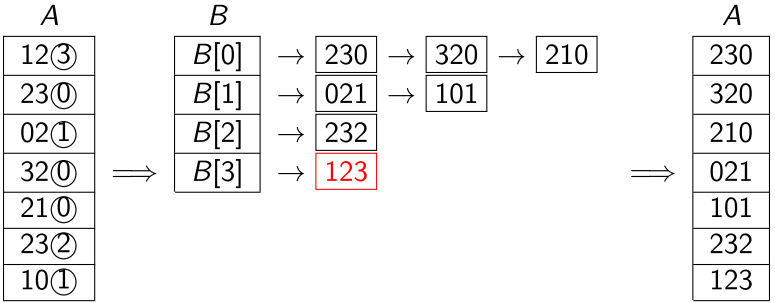

(Single-digit) *bucket-sort*

```
bucket-sort(A, n, sort-key(·))
A: array of size n
sort-key(·) : maps items of A to {0, ..., R−1}
Initialize an array B[0...R − 1] of empty queues (buckets)
for i ← 0 to n−1 do
        Append A[i] at end of B[sort-key(A[i])]
i ← 0
for j ← 0 to R − 1 do
        while B[j] is non-empty do
                move front element of B[j] to A[i++]
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- In our example *sort-key*$(A[i])$ returns the last digit of $A[i]$

(Single-digit) *bucket-sort*

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- In our example *sort-key*$(A[i])$ returns the last digit of $A[i]$
- *bucket-sort* is **stable**: equal items stay in original order.
- Run-time $\Theta(n + R)$, auxiliary space $\Theta(n + R)$
- It is possible to replace the lists by arrays $\rightsquigarrow$ *count-sort* (no details).

Most-significant-digit(MSD)-radix-sort

Sort array of m-digit radix-R numbers recursively:
sort by 1st digit, then each group by 2nd digit, etc.

①23
②32
⓪21
③20
②10
②30
①01

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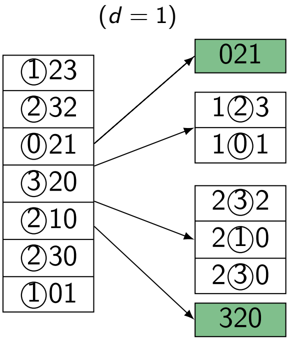

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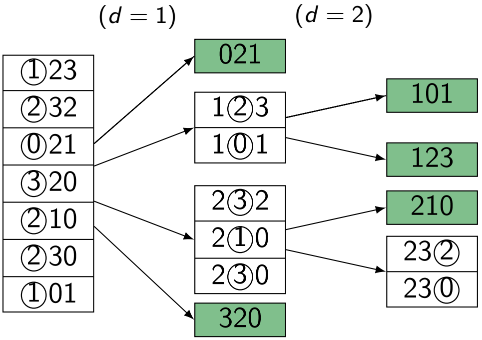

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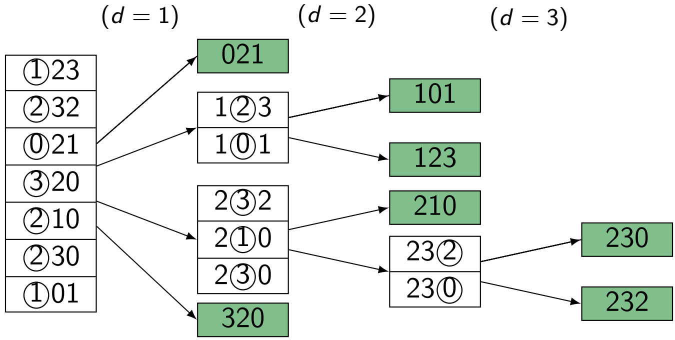

MSD-radix-sort

MSD-radix-sort$(A, n, \quad d \leftarrow 1)$
A: array of size n, contains m-digit radix-R numbers
if $(d \leq m$ and $(n > 1))$
    bucket-sort(A, n,‘return dth digit of $A[i]$’)
    $\ell \leftarrow 0$ `// find sub-arrays and recurse`
    for $j \leftarrow 0$ to $R - 1$
        Let $r \geq \ell - 1$ be maximal s.t. $A[\ell..r]$ have dth digit j
        MSD-radix-sort$(A[\ell..r], r{-}\ell{+}1, d{+}1)$
        $\ell \leftarrow r + 1$

Analysis:

- $\Theta(m)$ levels of recursion in worst-case.
- $\Theta(n)$ subproblems on most levels in worst-case.
- $\Theta(R + (\text{size of sub-array}))$ time for each *bucket-sort* call.

$\Rightarrow$ Run-time $\Theta(mnR)$ — slow. Many recursions and allocated arrays.

Least-significant-digit(LSD)-radix-sort

LSD-radix-sort(A, n)
A: array of size n, contains m-digit radix-R numbers
1. **for** $d \leftarrow$ least significant to most significant digit **do**
2. *bucket-sort*$(A$, n, 'return dth digit of $A[i]$')

12③		2③0		①01		021
23⓪		3②0		②10		101
02①	$(d=3)$	2①0	$(d=2)$	③20	$(d=1)$	123
32⓪	$\Longrightarrow$	0②1	$\Longrightarrow$	⓪21	$\Longrightarrow$	210
21⓪		1⓪1		①23		230
23②		2③2		②30		232
10①		1②3		②32		320

- Loop-invariant: A is sorted w.r.t. digits $d, \ldots, m$ of each entry.
- **Time cost**: $\Theta(m(n+R))$ **Auxiliary space**: $\Theta(n+R)$

Summary

- SORTING is an important and *very* well-studied problem
- Can be done in $\Theta(n \log n)$ time; faster is not possible for general input
- *heap-sort* is the only $\Theta(n \log n)$-time algorithm we have seen with $O(1)$ auxiliary space.
- *merge-sort* is also $\Theta(n \log n)$, selection & insertion sorts are $\Theta(n^2)$.
- *quick-sort* is worst-case $\Theta(n^2)$, but often the fastest in practice
- *bucket-sort* and *radix-sort* achieve $o(n \log n)$ if the input is special

- Randomized algorithms can eliminate "bad cases"
- Best-case, worst-case, average-case can all differ.
- Often it is easier to analyze the run-time on randomly chosen input rather than the average-case run-time.