CS 240 – Data Structures and Data Management

Module 2: Priority Queues

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Based on lecture notes by many previous cs240 instructors

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Outline

2 Priority Queues
- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Binary Heaps as PQ realization
- *PQ-sort* and *heap-sort*
- Towards the Selection Problem

Outline

Abstract Data Types (review)

Abstract Data Type (ADT): A description of *information* and a collection of *operations* on that information.

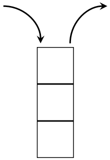
The information is accessed *only* through the operations.

We can have various **realizations** of an ADT, which specify:

- How the information is stored (**data structure**)
- How the operations are performed (**algorithms**)

ADT Stack (review)

Stack: an ADT consisting of a collection of items with operations:



- *push*: Add an item to the stack.
- *pop*: Remove and return the most recently added item.

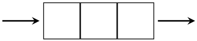
Items are removed in LIFO (*last-in first-out*) order.

We can have extra operations: *size*, *is-empty*, and *top*

ADT Stack can easily be realized using arrays or linked lists such that operations take constant time, up to resizing arrays (exercise).

ADT Queue (review)

Queue: an ADT consisting of a collection of items with operations:



- *enqueue* (or *append* or *add-back*): Add an item to the queue.
- *dequeue* (or *remove-front*): Remove and return the least recently inserted item.

Items are removed in FIFO (*first-in first-out*) order.

We can have extra operations: *size*, *is-empty*, and *peek*/*front*

ADT Queue can easily be realized using (circular) arrays or linked lists such that operations take constant time, up to resizing arrays (exercise).

Outline

ADT Priority Queue

Priority Queue generalizes both ADT Stack and ADT Queue.

It is a collection of items (each having a **priority** or **key**) with operations

- *insert*: inserting an item tagged with a priority
- *delete-max*: removing and returning an item of *highest* priority.

We can have extra operations: *size*, *is-empty*, and *get-max*

This is a **maximum-oriented** priority queue. A **minimum-oriented** priority queue replaces operation *delete-max* by *delete-min*.

Applications:

- How would you simulate a stack with a priority queue?
- How would you simulate a queue with a priority queue?
- Other applications: typical todo-list, simulation systems, sorting

Using a Priority Queue to Sort

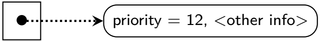
PQ-Sort($A[0..n-1]$)
1. initialize PQ to an empty priority queue
2. **for** $i \leftarrow 0$ **to** $n-1$ **do**
3. $\quad PQ.$*insert*(an item with priority $A[i]$)
4. **for** $i \leftarrow n-1$ **down to** 0 **do**
5. $\quad A[i] \leftarrow$ priority of $PQ.$*delete-max*()

- Note: run-time depends on how we implement the priority queue.
- Sometimes written as: $O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{delete-max})$

Realizations of Priority Queues

0	1	2	3	4
12	99	37		

Realization 1: unsorted arrays

(In our examples we only show the priorities, and we show them directly in the node. A more accurate picture would be 

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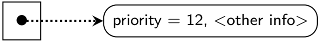
(In our examples we only show the priorities, and we show them directly in the node. A more accurate picture would be 

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Run-time of operations:

- *insert*: $\Theta(1)$
- *delete-max*: $\Theta(n)$

Note: We assume **dynamic arrays**, i. e., expand by doubling as needed. (Amortized over all insertions this takes $\Theta(1)$ extra time.)

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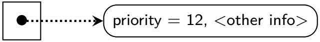
(In our examples we only show the priorities, and we show them directly in the node. A more accurate picture would be 

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PQ-sort with this realization yields *selection-sort*.
Using unsorted linked lists is identical.

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Realizations of Priority Queues

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Run-time of operations:

- *insert*: $\Theta(n)$
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PQ-sort with this realization yields *insertion-sort*.
Using sorted linked lists is identical.

Main advantage:

- *insert* can be implemented to have $\Theta(1)$ best-case run-time (how?)
- *insertion-sort* then has $\Theta(n)$ best-case run-time

Outline

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   - Abstract Data Types
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   - Binary Heaps
   - Binary Heaps as PQ realization
   - *PQ-sort* and *heap-sort*
   - Towards the Selection Problem

Towards Realization 3: Heaps

A **(binary) heap** is a certain type of binary tree.

You should know:

- A **binary tree** is either
  - empty, or
  - consists of three parts:
    a node and two binary trees (left subtree and right subtree).
- Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc.
- **Level** ℓ = all nodes with distance ℓ from the root. Root is on level 0.
- **Height** h = maximum number for which level h contains nodes. Single-node tree has height 0, empty tree has height -1.
- Known: Any binary tree with height h has $n \leq 2^{h+1} - 1$ nodes.

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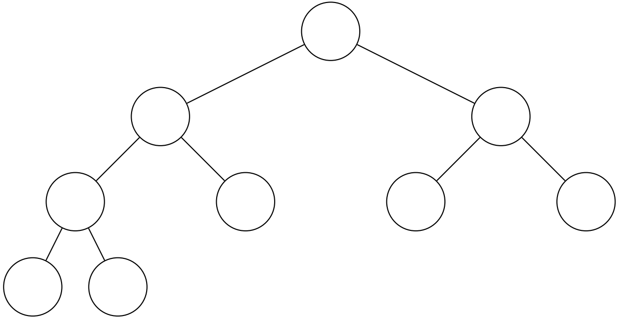
Example Binary Tree and Heap



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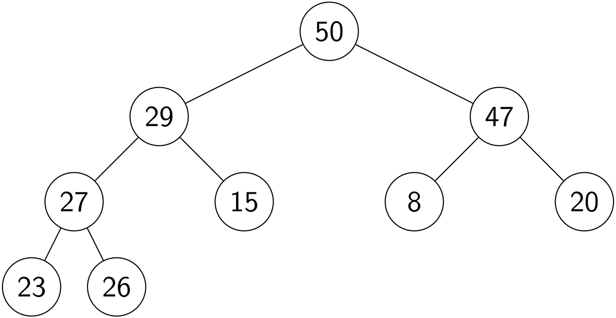
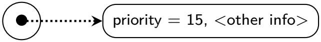


Binary tree with

1. structural property and
2. heap-order property.

Recall: 

represents



Heaps – Definition

A **heap** is a binary tree with the following two properties:

1. **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.

2. **Heap-order Property:** For any node i, the key of the parent of i is larger than or equal to key of i.

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Lemma: The height of a heap with n nodes is $\Theta(\log n)$.

Storing Heaps in Arrays

Heaps should *not* be stored as binary trees!

Let H be a heap of n items and let A be an array of size n. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

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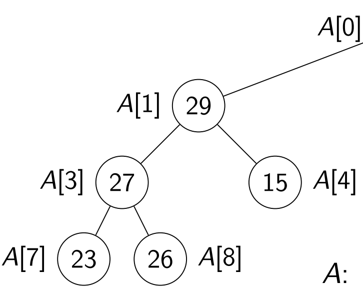

	0	1	2	3	4	5	6	7	8
A:	50	29	47	27	15	8	20	23	26

Heaps in Arrays – Navigation

It is easy to navigate the heap using this array representation:

- the *root* node is at index 0
  (We use "node" and "index" interchangeably in this implementation.)
- the *last* node is $n - 1$ (where n is the size)
- the *left child* of node i (if it exists) is node $2i + 1$
- the *right child* of node i (if it exists) is node $2i + 2$
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We should hide implementation details using helper-functions!

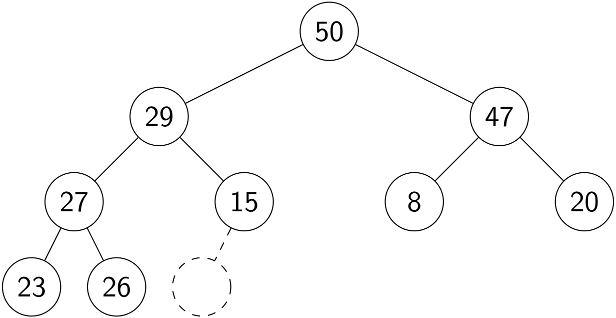
- functions *root()*, *last()*, *parent(i)*, etc.

Some of these helper-functions need to know the size n. We assume that the data structure stores this explicitly.

Outline

insert in Heaps

insert(35):



- By structural property: no choice where the new node can go.

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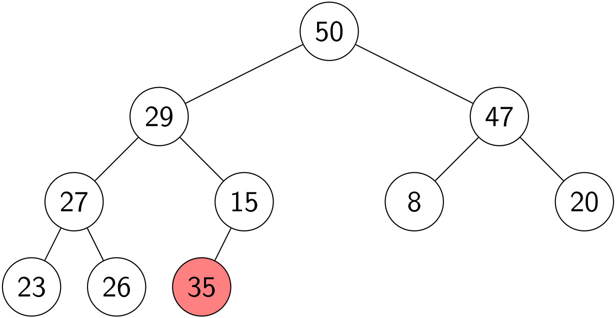

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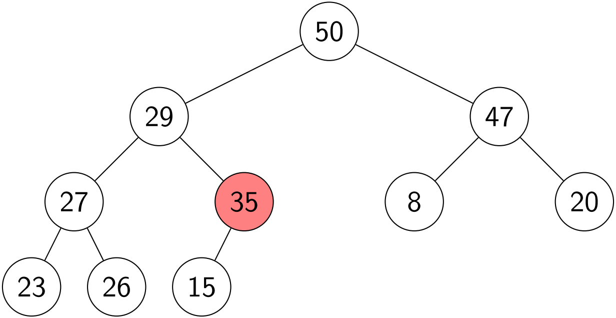

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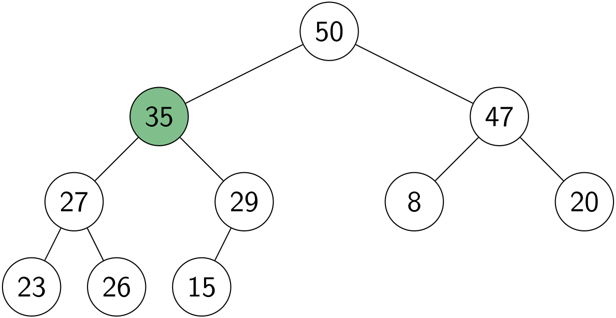

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Insert in Heaps

- Place the new key at the first free leaf
- Use *fix-up* to restore heap-order.

insert(x)
1. $\ell \leftarrow$ *last*()$+1$
2. $A[\ell] \leftarrow x$ `// assume dynamic array used`
3. increase *size* `//` *size*`: stored by PQ`
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fix-up(A, i)
i: an index corresponding to a node of the heap
1. **while** $parent(i)$ exists **and** $A[parent(i)].key < A[i].key$ **do**
2. swap $A[i]$ and $A[parent(i)]$
3. $i \leftarrow parent(i)$

Insert in Heaps

- Place the new key at the first free leaf
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```
insert(x)
    ℓ ← last()+1
    A[ℓ] ← x               // assume dynamic array used
    increase size          // size:  stored by PQ
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fix-up(A, i)
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        swap A[i] and A[parent(i)]
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```

Time: $O(\text{height of heap}) = O(\log n)$ (and this is tight).

(Correctness may seem obvious, but is actually non-trivial.)

delete-max in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a *fix-down*:

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```
fix-down(A, i)
A: an array that stores a heap of size n
i: an index corresponding to a node of the heap
while i is not a leaf do
    j ← left child of i        // Find the child with the larger key
    if (i has right child and A[right child of i].key > A[j].key)
        j ← right child of i
    if A[i].key ≥ A[j].key break
    swap A[j] and A[i]
    i ← j
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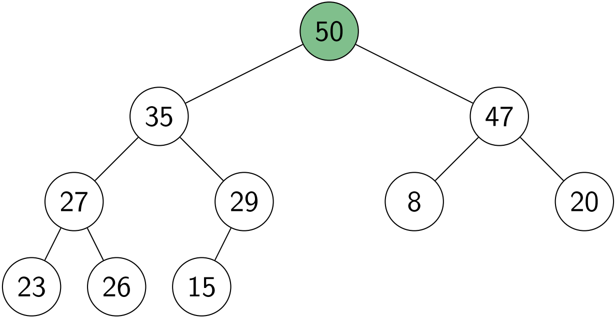
delete-max example



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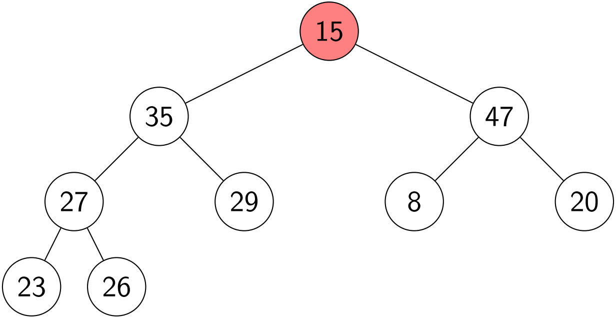

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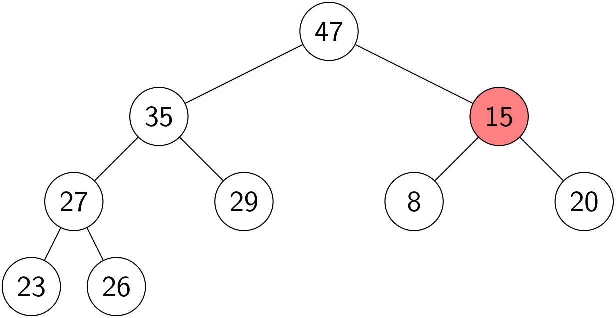

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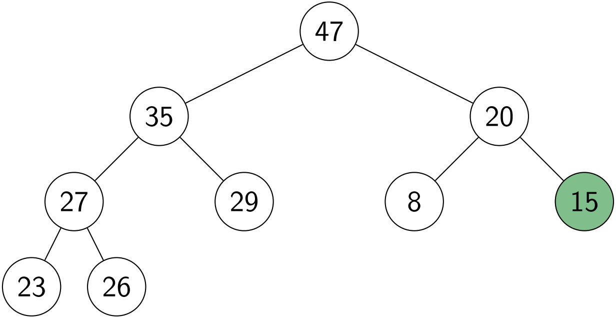

Priority Queue Realization Using Heaps

delete-max()
1. $\ell \leftarrow$ *last*()
2. *swap* A[*root*()] and $A[\ell]$
3. decrease *size*
4. *fix-down*(A, *root*(), *size*)
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Binary heap are a realization of priority queues where the operations *insert* and *delete-max* take $\Theta(\log n)$ **time**.

Outline

Sorting using heaps

- Recall: Any priority queue can be used to *sort* in time

$$O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{delete-max})$$

- Using the binary-heaps implementation of PQs, we obtain:

```
PQ-sort-with-heaps(A)
1.  initialize H to an empty heap
2.  for i ← 0 to n − 1 do
3.      H.insert(A[i])
4.  for i ← n − 1 down to 0 do
5.      A[i] ← H.delete-max()
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⇝ *PQ-sort* using heaps takes $O(n \log n)$ time (and this is tight).

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⇝ *PQ-sort* using heaps takes $O(n \log n)$ time (and this is tight).

- Can improve this with two simple tricks → **heap-sort**
  1. Can use the same array for input and heap. ⇝ *$O(1)$ auxiliary space!*
  2. Heaps can be built faster if we know all input in advance.

Building Heaps with *fix-up*

Problem: Given n items all at once (in $A[0 \cdots n-1]$) build a heap containing all of them.

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Solution 1: Start with an empty heap and insert items one at a time:

```
simple-heap-building(A)
A: an array
initialize H as an empty heap
for i ← 0 to A.size() − 1 do
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This corresponds to doing *fix-ups*
Worst-case running time: $O(n \log n)$ (and this is tight).

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Solution 2: Using *fix-downs* instead:

heapify(A)
A: an array
1. $n \leftarrow A.size()$
2. **for** $i \leftarrow$ *parent*(*last*()) **downto** *root*() **do**
3. *fix-down*(A, i, n)

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1.  n ← A.size()
2.  for i ← parent(last()) downto root() do
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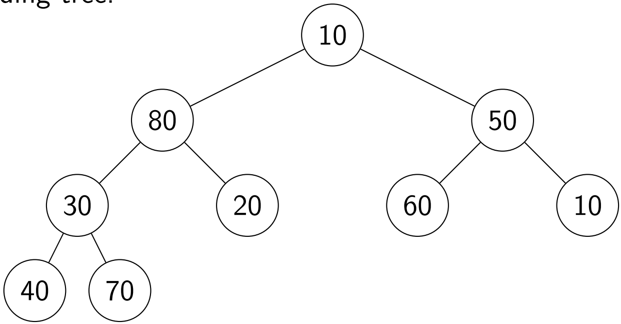
A careful analysis yields a worst-case complexity of $\Theta(n)$.
A heap can be built in linear time.

heapify example

A :

0	1	2	3	4	5	6	7	8
10	80	50	30	20	60	10	40	70

Corresponding tree:



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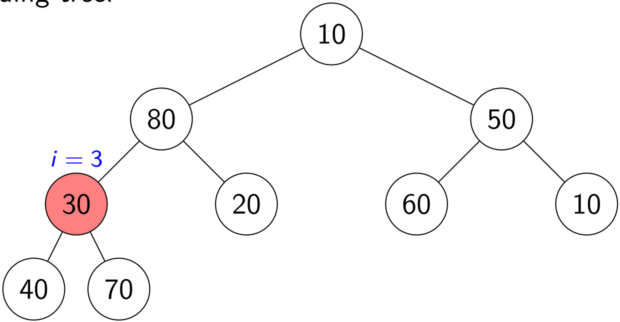

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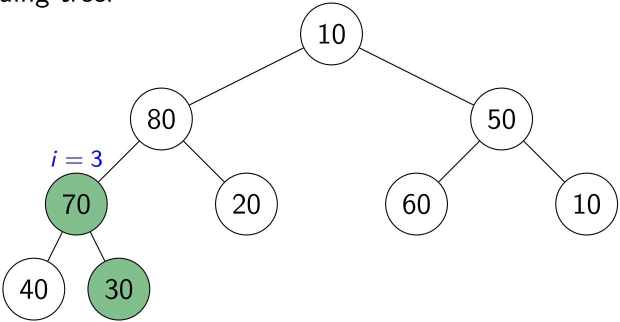

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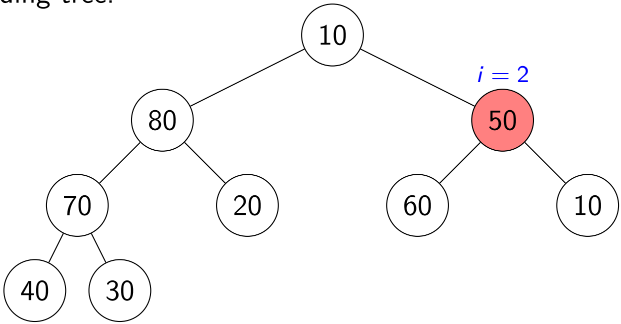

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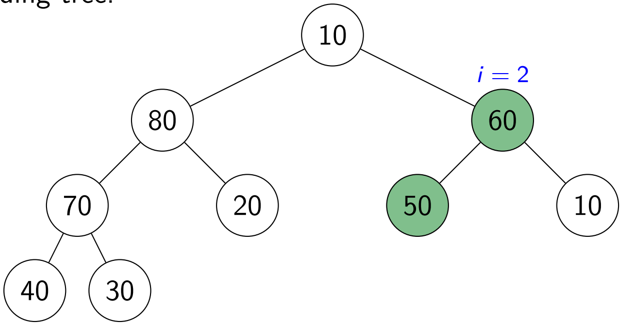

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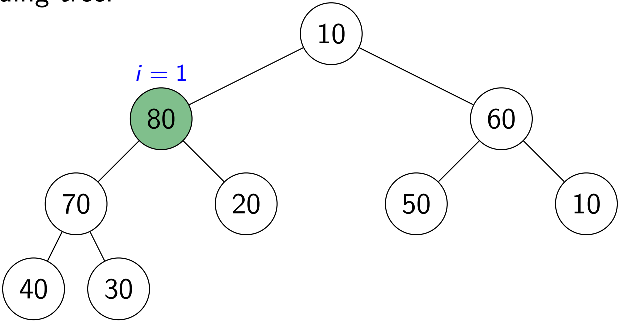
	10	80	60	70	20	10	40	30

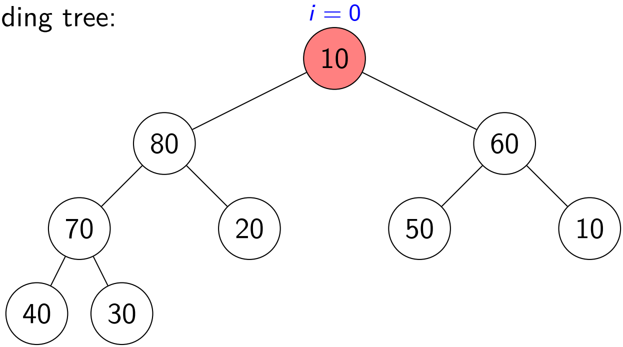
Corresponding tree:



heapify example

	0	1	2	3	4	6	7	8
A :	10	80	60	70	20	10	40	30

Corresponding tree:



heapify example

	0	1	2	3	4	6	7	8
A :	80	10	60	70	20	10	40	30

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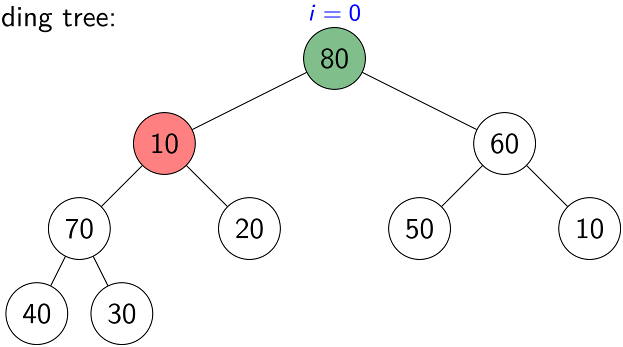

heapify example

	0	1	2	3	4	6	7	8
A :	80	70	60	10	20	10	40	30

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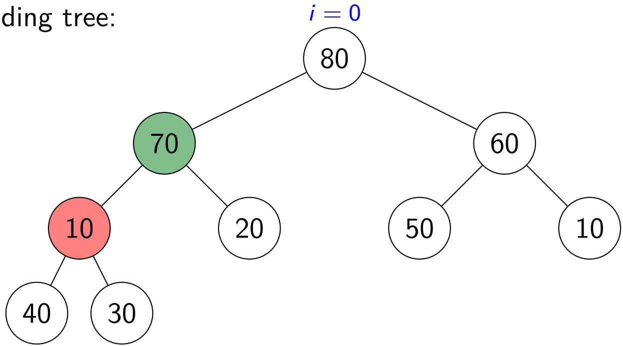

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	0	2	3	4	6	7	8
A :	80	60	40	20	10	10	30

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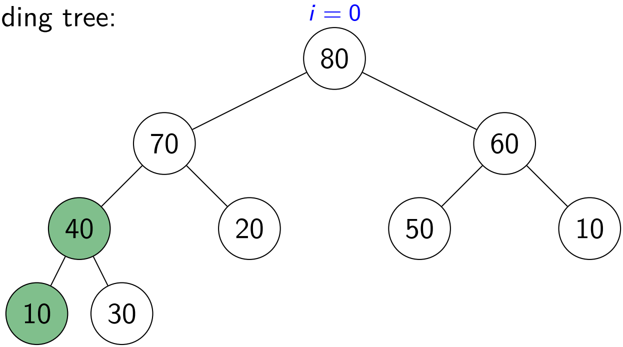


Efficient sorting with heaps

- Idea: *PQ-sort* with heaps.
- $O(1)$ auxiliary space: Use same input-array A for storing heap.

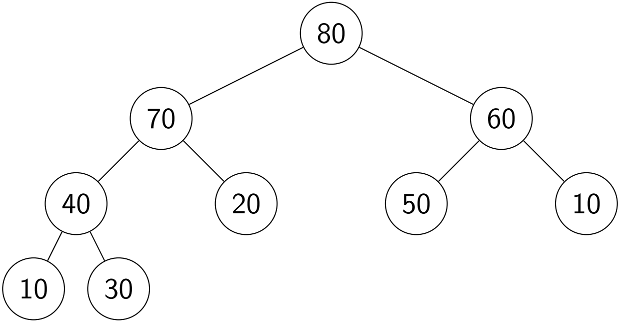
```
heap-sort(A)
// heapify
n ← A.size()
for i ← parent(last()) downto 0 do
        fix-down(A, i, n)

// repeatedly find maximum
while n > 1
        // 'delete' maximum by moving to end and decreasing n
        swap items at A[root()] and A[last()]
        decrease n
        fix-down(A, root(), n)
```

The for-loop takes $\Theta(n)$ time and the while-loop takes $\Theta(n \log n)$ time.

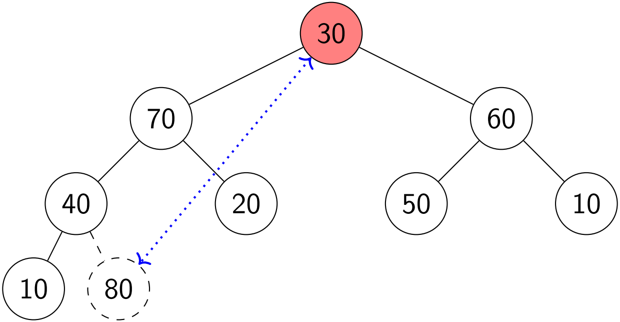
heap-sort example

Continue with the example from heapify:



heap-sort example

Continue with the example from heapify:



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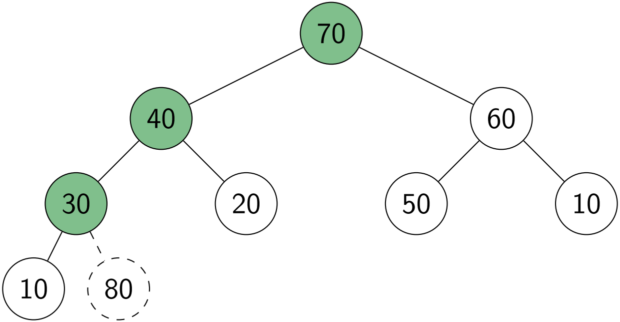

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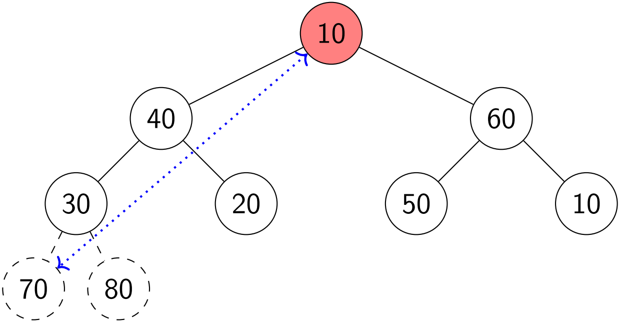

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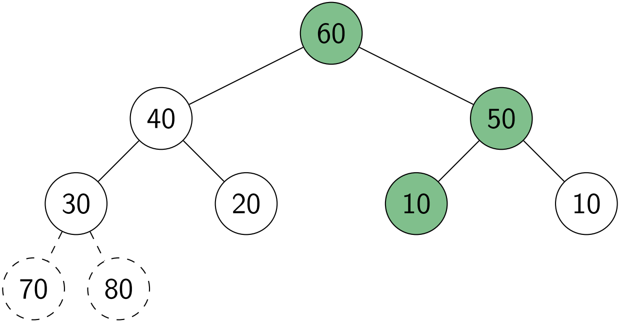

heap-sort example

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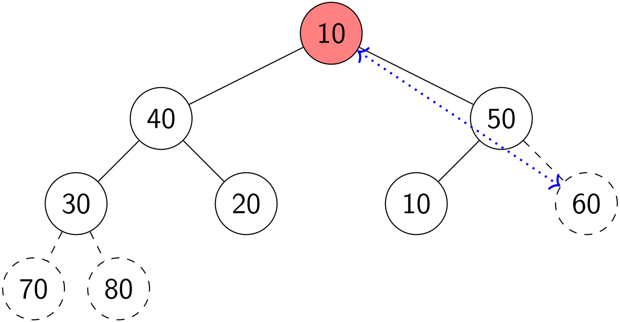

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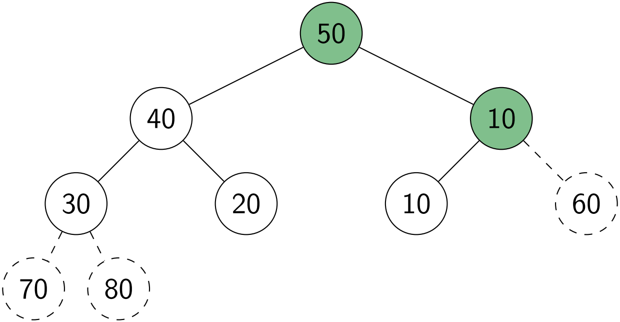

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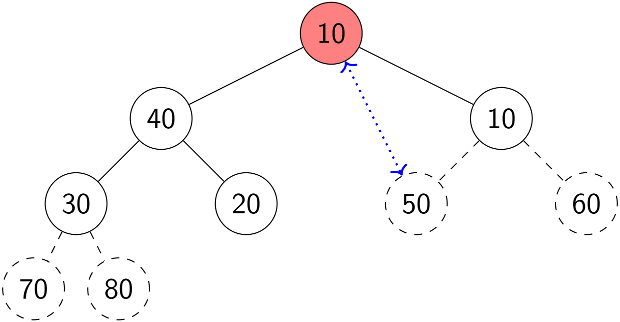

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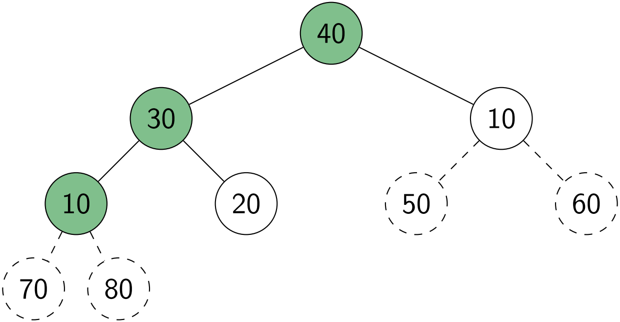

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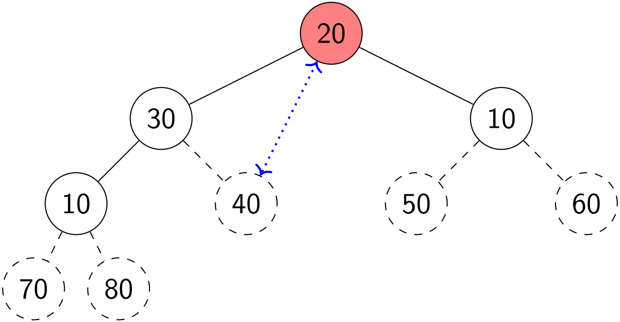

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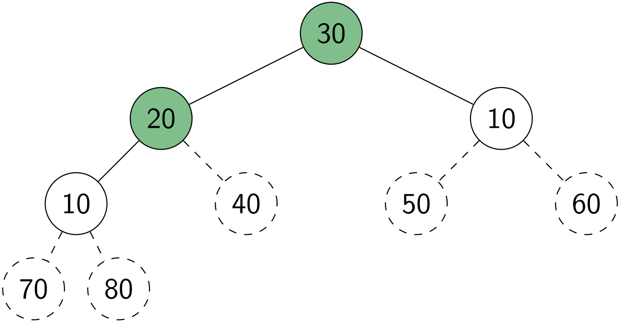

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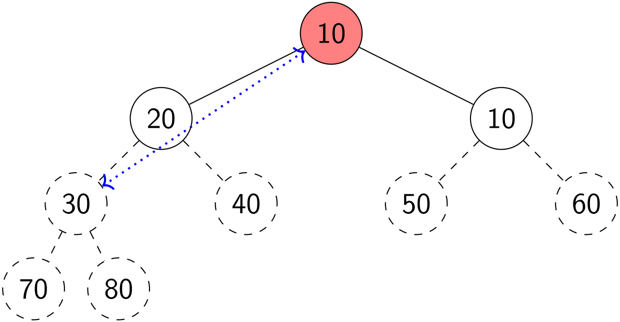

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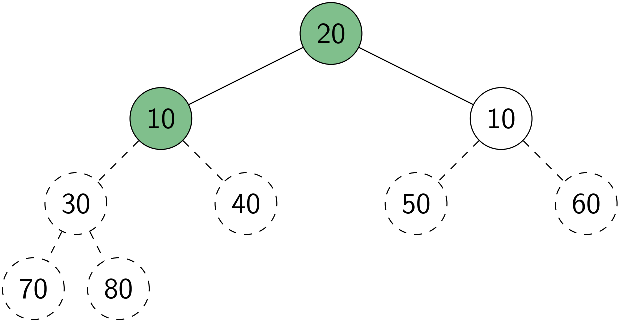

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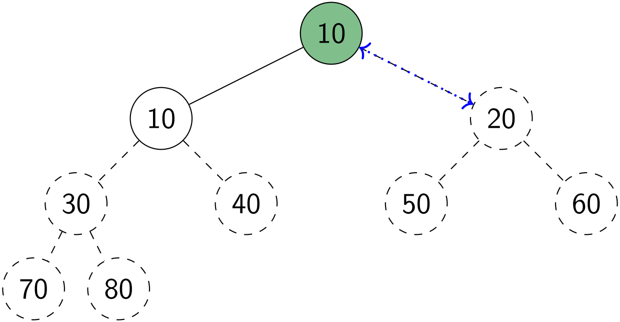

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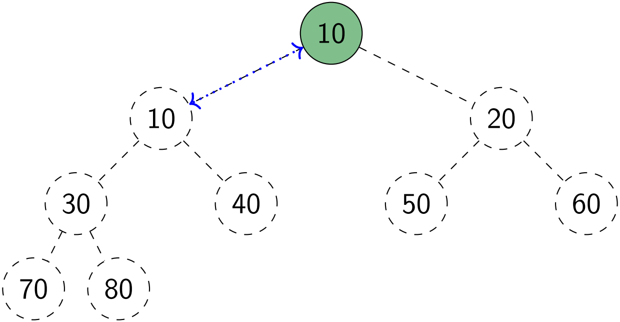

The array (i.e., the heap in level-by-level order) is now in sorted order.

Heap summary

- **Binary heap**: A binary tree that satisfies structural property and heap-order property.
- Heaps are one possible realization of ADT PriorityQueue:
  - *insert* takes time $O(\log n)$
  - *delete-max* takes time $O(\log n)$
  - Also supports *findMax* in time $O(1)$
- A binary heap can be built in linear time.
- *PQ-sort* with binary heaps leads to a sorting algorithm with $O(n \log n)$ worst-case run-time ($\rightsquigarrow$ *heap-sort*)
- We have seen here the *max-oriented version* of heaps (the maximum priority is at the root).
- There exists a symmetric *min-oriented version* that supports *insert* and *delete-min* with the same run-times.

Outline

2 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Binary Heaps as PQ realization
- *PQ-sort* and *heap-sort*
- Towards the Selection Problem

Finding the smallest items

Problem: Find the *kth smallest item* in an array A of n numbers.

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Complexity: $\Theta(kn)$.

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Complexity: $\Theta(n + k \log n)$.

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We can achieve $\Theta(n \log n)$ worst-case time easily, but can we do better?