University of Waterloo
CS240 Spring 2024
Assignment 1

Due Date: Tuesday, May 21 at 5:00pm

Please read the following link for guidelines on submission:

`https://student.cs.uwaterloo.ca/~cs240/w24/assignments.phtml#guidelines`

Each question must be submitted individually to MarkUs as a PDF with the corresponding file names: `a1q1.pdf`, `a1q2.pdf`, ... It is a good idea to submit questions as you go so you aren't trying to create several PDF files at the last minute.

Late Policy: Assignments are due at **5:00pm**, with the grace period until 11:59pm.

Notes:

- Logarithms are in base 2, if not mentioned otherwise.
- A positive function is a function that takes positive real values.

Problem 1 [3+3+3+3=12 marks]

Provide a complete proof of the following statements from first principles (i.e., using the original definitions of order notation).

a) $27n^7 + 17n^3 \log n + 2024$ is $O(n^9)$

b) $n^2(\log n)^{1.0001}$ is $\Omega(n^2)$

c) $\frac{n^2}{n+\log n}$ is $\Theta(n)$

d) n^n is $\omega(n^{20})$

Problem 2 [3+3+3=9 marks]

For each pair of the following functions, fill in the correct asymptotic notation among Θ, o, and ω in the statement $f(n) \in \sqcup(g(n))$. Prove the relationship using any relationship or technique that described in class.

a) $f(n) = n^2 + 27n \log n + 2024$ versus $g(n) = n^2 \log n + 2024$

b) $f(n) = 10^n + 99n^{10}$ versus $g(n) = 75^n + 25n^{27}$

c) $f(n) = \log \log n$ versus $g(n) = (\log \log \log n)^8$

Problem 3 [4+4+4+4=16 marks]

Prove or disprove each of the following statements. To prove a statement, you should provide a formal proof that is based on the definitions of the order notations. To disprove a statement, you can either provide a counter example and explain it or provide a formal proof. All functions are positive functions.

a) $f(n) \notin o(g(n))$ and $f(n) \notin \omega(g(n)) \Rightarrow f(n) \in \Theta(g(n))$

b) $f(n) \in \Theta(g(n))$ and $h(n) \in \Theta(g(n)) \Rightarrow \frac{f(n)}{h(n)} \in \Theta(1)$

c) $f(n) \in \Theta(g(n)) \Rightarrow 2^{f(n)} \in \Theta(2^{g(n)})$

d) $\min(f(n), g(n)) \in \Theta\left(\frac{f(n)g(n)}{f(n)+g(n)}\right)$

Problem 4 [4+4+4=12 marks]

Analyze the following piece of pseudocode and give a tight (Θ) bound on the running time as a function of n. Show your work. A formal proof is not required, but you should justify your answer (in all cases, n is assumed to be a positive integer).

a)

```
s = 0
for i = 1 to n do
   for j = i to n do
      if i == j then
         k = n
         while k > 0 do
            s = s + 1
            k = k/3
```

b)

```
s = 2024
for i = 1 to n*n do
   for j = 1 to i*i do
      s = s + 13
```

c)

```
s = 13
i = 1
while i < 10n do
   j = n^5  // n to the power 5
   while j > i do
      s = s + 7
      j = j - i
   i = i + 10
```

Problem 5 [2+4=6 marks]

Dr. I. M. Smart has invented a new class of functions, denoted $O'(f)$: A function $f(n)$ is in $O'(g)$ if there is a constant $c > 0$ such that $f(n) \leq cg(n)$ for all $n \geq 0$. Assume that all functions are defined on non-negative integers and take positive real values (i.e. the domain is non-negative integers and the range is positive reals).

a) Prove that $f(n) \in O'(g(n))$ implies that $f(n) \in O(g(n))$.

b) Prove that $f(n) \in O(g(n))$ implies that $f(n) \in O'(g(n))$.