07: Natural Numbers

We'll review how we derived the list template.

```
;; A (listof X) is one of:
;; ⋆ empty
;; ⋆ (cons X (listof X))
```

Suppose we have a list, `lst`.

- The test `(empty? lst)` tells us which case applies.
- If `(empty? lst)` is `false`, then `lst` is of the form `(cons f r)`.
  - `f` is `(first lst)`.
  - `r` is `(rest lst)`.

Because `r` is a list, we recursively apply the function we are constructing to it.

```
;; listof-X-template: (listof X) → Any
(define (listof-X-template lst)
  (cond [(empty? lst) ...]
        [else (... (first lst)
                   (listof-X-template (rest lst)))]))
```

We can repeat this reasoning on a recursive definition of **natural numbers** to obtain a template.

Logicians use the Peano axioms to define the natural numbers. These include:

- 0 is a natural number.
- For every natural number n, $S(n)$ is a natural number.

1 can be represented as $S(0)$, 2 as $S(S(0))$, 3 as $S(S(S(0)))$, and so on.

$S(n)$ is called the successor function; it consumes a natural number, and returns the next.

(A handful of other axioms define the rest of the behaviour of natural numbers, but we don't need to go into them here.)

A Formal Definition of Natural Numbers (2/3)

The successor function $S(n)$ produces the “next” natural number. We will use the Racket function `add1` as the successor function:

```
(add1 0) ⇒ 1
(add1 1) ⇒ 2
(add1 2) ⇒ 3
```

With this function, we can translate the logicians’ axioms into a Racket data definition:

- 0 is a natural number.
- For every natural number n, $S(n)$ is a natural number.

$\longrightarrow$

```
;; A Nat is one of:
;;   0
;;   (add1 Nat)
```

```
;; A Nat is one of:
;; ⋆ 0
;; ⋆ (add1 Nat)
```

The natural numbers start at 0 in computer science and some branches of mathematics (e.g., logic).

We’ll now work out a template for functions that consume a natural number.

Suppose we have a natural number, *n*. Then it must conform to our data definition:

```
;; A Nat is one of:
;; * 0
;; * (add1 Nat)
```

The test `(zero? n)` tells us which of these cases applies, yielding:

```
;; nat-template: Nat -> Any
(define (nat-template n)
  (cond [(zero? n) ...]  ;; n is 0
        [else ...]))     ;; n is (add1 k), for some k
```

We can compute *k* with `(- n 1)` or `(sub1 n)`.

Because *k* is a natural number, we recursively apply the function we are constructing to it.

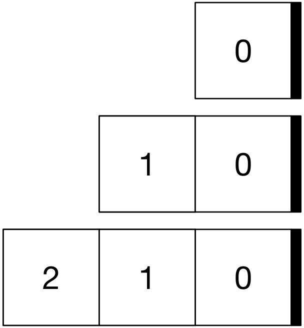
```
;; nat-template: Nat -> Any
(define (nat-template n)
  (cond [(zero? n) ...]
        [else (... n
                   (nat-template (sub1 n)))]))
```

Goal: `countdown`, which consumes a natural number *n* and produces a decreasing list of all natural numbers less than or equal to *n*.

```
(countdown 0) ⇒ (cons 0 empty)

(countdown 1) ⇒ (cons 1 (cons 0 empty))

(countdown 2) ⇒ (cons 2 (cons 1 (cons 0 empty)))
```



With these examples, we proceed by filling in the template.

```
;; (countdown n) produces a decreasing list of Nats from n to 0
(check-expect (countdown 0) (cons 0 empty))
(check-expect (countdown 2) (cons 2 (cons 1 (cons 0 empty))))

;; countdown: Nat → (listof Nat)
(define (countdown n)
  (cond [(zero? n) ...]
        [else (... n
                   (countdown (sub1 n)))]))
```

Useful questions:

1. What do we produce in the base case?
2. In the recursive case, what (if anything) do we do to transform `n`?
3. What is the result of processing `(f (sub1 n))` recursively?
4. How do we combine steps 2 and 3 to obtain the result for `(f n)`?

```
;; (countdown n) produces a decreasing list of Nats from n to 0
;; Examples:
(check-expect (countdown 0) (cons 0 empty))
(check-expect (countdown 2) (cons 2 (cons 1 (cons 0 empty))))

;; countdown: Nat → (listof Nat)
(define (countdown n)
  (cond [(zero? n) (cons 0 empty)]
        [else (cons n (countdown (sub1 n)))]))
```

```
(countdown 2)
⇒ (cons 2 (countdown 1))
⇒ (cons 2 (cons 1 (countdown 0)))
⇒ (cons 2 (cons 1 (cons 0 empty)))
```

Ex. 1

Write a recursive function `(sum-to n)` that consumes a `Nat` and produces the sum of all `Nat` between `0` and `n`.

`(sum-to 4)` $\Rightarrow$ `(+ 4 (+ 3 (+ 2 (+ 1 0))))` $\Rightarrow$ `10`

The symbol $\mathbb{Z}$ is often used to denote the integers.

We can add subscripts to define subsets of the integers (also known as **intervals**).

For example, $\mathbb{Z}_{\geq 0}$ defines the non-negative integers, also known as the natural numbers.

Other examples: $\mathbb{Z}_{>4}$, $\mathbb{Z}_{<-8}$, $\mathbb{Z}_{\leq 1}$.

If we change the base case test from `(zero? n)` to `(= n 7)`, we can stop the countdown at 7.

This corresponds to the following definition:

```
;; An integer in ℤ≥7 is one of:
;; ⋆ 7
;; ⋆ (add1 ℤ≥7)
```

We use this data definition as a guide when writing functions, but in practice we use a requires section in the contract to capture the new stopping point.

```
;; (countdown-to-7 n) produces a decreasing list from n to 7
```

Tracing `countdown-to-7`:

```
(countdown-to-7 9)
⇒ (cons 9 (countdown-to-7 8))
⇒ (cons 9 (cons 8 (countdown-to-7 7)))
⇒ (cons 9 (cons 8 (cons 7 empty)))
```

We can generalize both `countdown` and `countdown-to-7` by providing the base value (e.g., 0 or 7) as a second parameter `base`.

Here, the stopping condition will depend on `base`.

The parameter `base` has to **"go along for the ride"** (be passed unchanged) in the recursion.

```
;; (countdown-to n base) produces a decreasing list from n to base
;; Examples:
(check-expect (countdown-to 4 2) (cons 4 (cons 3 (cons 2 empty))))
(check-expect (countdown-to 7 7) (cons 7 empty))

;; countdown-to: Int Int → (listof Int)
;; Requires: n >= base
(define (countdown-to n base)
  (cond [(= n base) (cons base empty)]
        [else (cons n (countdown-to (sub1 n) base))]))
```

```
(countdown-to 4 2)
⇒ (cons 4 (countdown-to 3 2))
⇒ (cons 4 (cons 3 (countdown-to 2 2)))
⇒ (cons 4 (cons 3 (cons 2 empty)))
```

`countdown-to` works just fine if we put in negative numbers.

```
(countdown-to 1 -2)
⇒ (cons 1 (cons 0 (cons -1 (cons -2 empty))))
```

Ex. 2

Write a recursive function `(sum-between n b)` than consumes two `Nat`, with `n` $\geq$ `b`, and returns the sum of all `Nat` between `b` and `n`.

`(sum-between 5 3)` ⇒ `(+ 5 (+ 4 3))` ⇒ `12`

What if we want an increasing count?

Consider the non-positive integers $\mathbb{Z}_{\leq 0}$.

```
;; A integer in ℤ≤0 is one of:
;; ⋆ 0
;; ⋆ (sub1 ℤ≤0)
```

Examples: -1 is `(sub1 0)`, -2 is `(sub1 (sub1 0))`.

Since `(add1 (sub1 n)) => n` for all integers `n`, the inverse function we need is `add1`.

This suggests the following template.

Notice the additional requires section.

```
;; nonpos-template: Int → Any
;; Requires: n ≤ 0
(define (nonpos-template n)
  (cond [(zero? n) ...]
        [else (... n
                   (nonpos-template (add1 n)))]))
```

We can use this to develop a function to produce lists such as `(cons -2 (cons -1 (cons 0 empty)))`.

```
;; (countup n) produces an increasing list from n to 0
;; Example:
(check-expect (countup -2) (cons -2 (cons -1 (cons 0 empty))))

;; countup: Int → (listof Int)
;; Requires: n <= 0
(define (countup n)
  (cond [(zero? n) (cons 0 empty)]
        [else (cons n (countup (add1 n)))]))
```

As before, we can generalize this to counting up to *b*, by introducing `base` as a second parameter in a template.

```
;; (countup-to n base) produces an increasing list from n to base
;; Example:
(check-expect (countup-to 6 8) (cons 6 (cons 7 (cons 8 empty))))

;; countup-to: Int Int → (listof Int)
;; Requires: n <= base
(define (countup-to n base)
  (cond [(= n base) (cons base empty)]
        [else (cons n (countup-to (add1 n) base))]))
```

Example: `power`

The countdown/countup pattern is not only applicable to building lists. Consider calculating n^e where e, the exponent, is an integer.

The key insight is that $n^e = n * n^{e-1}$ and that n^0 is 1.

```
(check-expect (power 2 0) 1)
(check-expect (power 2 1) 2)
(check-expect (power 3 3) 27)
```

```
;; nat-template: Nat -> Any
(define (nat-template n)
  (cond [(zero? n) ...]
        [else (... n
                   (nat-template (sub1 n)))]))
```

Example: power

With renaming, documentation, and adding parameters:

```
;; (power n e) computes n^e
;; power: Int Nat -> Int
(define (power n e)
  (cond [(zero? e) ...]
        [else (... n e
                   (power n (sub1 e)))]))
```

```
;; (power n e) computes n^e
;; power: Int Nat -> Int
(define (power n e)
  (cond [(zero? e) 1]
        [else (* n (power n (sub1 e)))]))
```

> Repetition in other languages

Many imperative programming languages offer several language constructs to do repetition:

```
for i = 1 to 10 do { ...  }
```

Racket offers one construct – recursion – that is flexible enough to handle these situations and more.

We will soon see how to use Racket's abstraction capabilities to abbreviate many common uses of recursion.

When you are learning to use recursion, sometimes you will "get it backwards" and use the countdown pattern when you should be using the countup pattern, or vice-versa.

If you're building a list and get it backwards, avoid using the built-in list function `reverse` to fix your error. It cannot always save a computation done in the wrong order.

Instead, learn to fix your mistake by using the right pattern.

You may **not** use `reverse` on assignments unless we say otherwise. You may not implement your own version, either.

Ex. 3

Write a function `(countdown-by top step)` that returns a `listof Nat` so the first is `top`, the next is `step` less, and so on, until the next one would be zero or less.

`(countdown-by 12 3)` ⇒ `(cons 12 (cons 9 (cons 6 (cons 3 empty))))`

`(countdown-by 11 3)` ⇒ `(cons 11 (cons 8 (cons 5 (cons 2 empty))))`

Consider: how must you change the base case of the template?

Ex. 4

This exercise recurses on a list and a `Nat` at the same time.

Complete `n-th-item`.

```
;; (n-th-item lst n) Produce the n-th item in lst, where (first lst) is
;;    the 0th.
;; Example:
(check-expect (n-th-item (cons 3 (cons 7 (cons 31 (cons 63 empty)))) 0) 3)
(check-expect (n-th-item (cons 3 (cons 7 (cons 31 (cons 63 empty)))) 3) 63)

;; n-th-item: (listof Any) Nat → Any
;; Requires: n < (length lst)
(define (n-th-item lst n) ...)
```

- You should understand the recursive definition of a natural number, and how it leads to a template for recursive functions that consume natural numbers.
- You should understand how subsets of the integers greater than or equal to some bound m, or less than or equal to such a bound, can be defined recursively, and how this leads to a template for recursive functions that "count down" or "count up". You should be able to write such functions.

The following functions and special forms have been introduced in this module:

`add1 sub1`

You should complete all exercises and assignments using only these and the functions and special forms introduced in earlier modules. The complete list is:

```
* + - ... / < <= = > >= abs add1 and boolean? ceiling char-alphabetic? char-downcase
    char-lower-case? char-numeric? char-upcase char-upper-case? char-whitespace? char<=?
    char<? char=? char>=? char>? char? check-error check-expect check-within cond cons
    cons? cos define define-struct define/trace e else empty? error even? exp expt first
    floor integer? length list->string list? log max min modulo negative? not
    number->string number? odd? or pi positive? quotient remainder rest round sgn sin sqr
    sqrt string->list string-append string-downcase string-length string-lower-case?
    string-numeric? string-upcase string-upper-case? string<=? string<? string=? string>=?
    string>? string? sub1 substring symbol=? symbol? tan zero?
```